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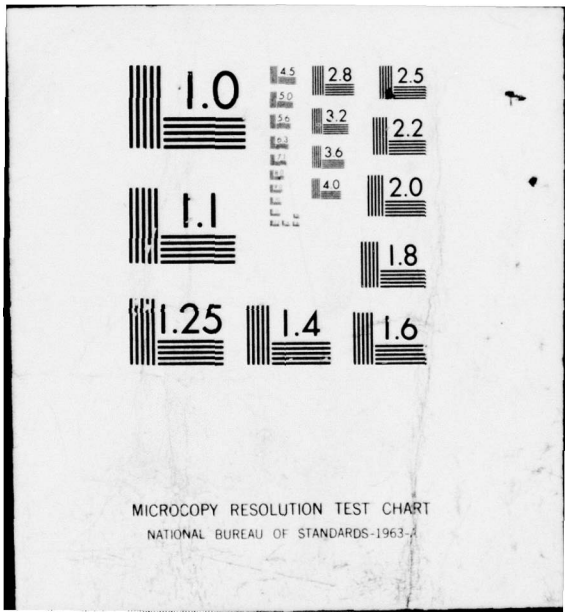
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November 1976



# A Hierarchical Technique for Mechanical Theorem Proving and Its Application to Programming Language Design

Norman Rubin



Courant Institute of Mathematical Sciences

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New York University



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## Acknowledgements

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# CHAPTER 1

## GENERAL INTRODUCTION AND REVIEW OF THE LITERATURE

The development of effective computer algorithms for general problem solving has long been considered one of the fundamental goals of Artificial Intelligence research. Within the AI community there appear to be two distinct points of view concerning the design of such algorithms. The first or "formalistic" view is based on the claim that problem solving is a special case of theorem proving. Here, a problem solving system begins by formulating the problem as the axioms of a logical system. Next a mechanical theorem prover is applied to produce a proof of the existence of a solution to the problem. Usually the techniques used to produce the proof are independent of the particular problem area. Finally the proof is translated back into a solution of the original problem.

This methodology for general problem solving is quite attractive. The development of algorithms may concentrate on the formal theorem prover rather than on the specific problem area, so each advance may contribute to problem solving as a whole. Theorem proving may be analyzed mathematically in terms of consistency, completeness, and adequacy. Eventually we may be able to compare the efficiency of two algorithms without comparing the efficiency of two implementations. Since the descriptions of these algorithms are precise mathematical statements, they can be easily communicated. Finally the meta-theorems which may be proved

are often elegant and interesting in their own right.

However, such an approach has so far proved to be extremely inefficient. Among the reasons for this are the following: Theorem provers do not use higher level planning, and their search procedures seem to lack structure. They do not make use of specialized information about the problem. Indeed, the very generality of uniform procedures seems to imply that information about specific problem domains cannot be included. Theorem provers almost always use a poor notation, such as, for example, n applications of the successor function to represent n. While we may formalize any fact within predicate calculus it is more difficult to formalize how to use that fact. Such information, which is sometimes called procedural knowledge, often cannot be given to a theorem prover at all. Finally theorem provers tend to get "lost" in what are minor aspects of a problem. For example most theorem provers find it difficult to show that 3 times 2 is 6; worse still, they find it more difficult to show that 3 times 2 is not 7.

The other point of view among AI researchers is called "intuitive". Here programs are designed to conform to ideas about how people solve particular problems. Usually these problems come from some toy domain. These intuitive systems are strong exactly where the formalistic systems are weak. That is they are efficient but obscure. For example, it is clear that Winograd's blocks world system [44] is an effective program; yet there is no simple explanation of exactly how it works.

Our own approach to problem solving has been between these two extremes but closer to the first. We began with a new theory of formal deduction which is based on a hierarchical system of cooperating theorem provers. Using a special case of this theory we developed a computer program which appears superior, on an initial set of test problems, to resolution-based schemes; we then extended the theorem proving system in two directions. The first of these, partial evaluation, allowed procedural knowledge, heuristics, and domain-dependent information to be given to the theorem prover. The second extension, completions, is a way of summarizing the partial proofs developed by a theorem prover with an eye toward producing useful lemmas. The theorem prover together with these extensions serves as the interpreter for a programming language, which we call Dilemma. This language is of very high level. The general objective underlying the design of Dilemma has been to provide a programming language in which all of the diverse heuristics of successful AI programs may be used in conjunction with the naturalness and clarity of the predicate calculus. Finally we wrote a program in Dilemma, which produces proofs for a small subset of geometry theorems. Experimental results show that this program can solve most of the geometry problems solved by intuitive AI systems; yet the program is much shorter.

Our experience suggests that the techniques used by Dilemma significantly extend the range of application of resolution-based systems. In some cases these techniques give a realistic alternative to the more ad hoc intuitive programs, with the advantage that programs written in Dilemma are much easier to write and to read and yet not much more time consuming for a machine to execute.

The remainder of this chapter is a historical review designed to show how our work is related to previous research. Chapter 2 develops the theorem proving system mathematically. Chapter 3 concentrates on the language. Chapter 4 describes the geometry program. Chapter 5 gives the experimental results which we obtained with the geometry program. Finally Chapter 6 gives our conclusions and some ideas on future work.

## 1.1 Declarative vs. Imperative Languages

In 1958, John McCarthy [25] proposed writing a program for solving problems by manipulating sentences in a formal language. He wished to have a program which would learn in a way similar to the way humans do. McCarthy argued that, in order for a program to be capable of learning something it must be capable of being told about it. Therefore, interesting ideas must be expressible in simple ways. To focus on simple ways to express ideas he looked at the difference between the form of language used to teach people and that used to teach machines. The difference was: a machine is instructed mainly in the form of a sequence of imperative sentences while a person is taught mainly in declarative sentences describing the situation in which action is required together with a few imperatives. Some tasks are given to people in the first form,e.g. cooking directions, how to build a model airplane, how to knit. However, complex tasks such as how to prove a theorem of mathematics are taught declaratively. McCarthy's distinction between declarative and imperative languages is basic to our work. Essentially, our view is that declarative languages do not contain flow-of-control statements. They contain definitions, concepts and facts. Any of these may be applied at any place. On the other hand, imperative languages have DO loops, GO TOs, WHILEs, and other control dictions, all of which serve to specify what to do next.

### 1.1.1 Advantages of Declarative Languages

(1) Previous knowledge can be used. A machine given a declarative sentence can integrate it into all the previous declarations.

(2) Declarative sentences have logical consequences. Ideally, the machine could automatically deduce for itself the consequences of what it already knows.

(3) Since the declarative sentences may be accessed in any order their meaning is much less dependent on the order that they are given to a machine. This can facilitate later improvements (or afterthoughts).

(4) The effects of a new declarative are less dependent on the previous state of the system. When a change is made to an imperative system it generally requires much detailed knowledge about the system. In declaratives, changes may be made knowing less.

(5) Declarative statements are easier for people to understand. A long program in an imperative form is usually difficult for a person to follow. He must first learn the global flow of the program and then the detailed interactions of each part. Parts of declarative programs on the other hand may be understood in isolation.

### 1.1.2 Advantages of Imperative Languages

(1) Imperative programs are faster, since the flow of control is explicit and no search is required.

(2) Computers execute imperative instructions. The problem of writing a compiler or interpreter is simplified since the source and target languages are of similar form.

(3) When it is clear how to do something, it is easier to state it as an imperative; when we try to encode a sequence of commands as declaratives we have to use unnatural tricky constructions, such as state variables.

(4) The search strategies can be explicit in imperative forms. If we do not know the correct search strategy then we do not really know how to solve the problem. In this case the use of declaratives just hides our ignorance.

## 1.2 Deductive Languages

The fact that imperative programs are faster has often been considered to outweigh all the advantages of declarative programs. Almost all computer languages have been imperative. Minsky [26] in his review of AI felt that languages of the type we are calling declarative could best be called "descriptive". This was based on the view that each program statement would represent a high level abstract description of a situation. Hewitt [20] offered the name "procedural" for the class of languages we call imperative.

He claimed that knowlege could best be represented in a machine in the form of statements of a language which could be directly executed, e.g., a definition would be a list of properties and the code to actually test if an item had these properties.

Some three years after his first paper, McCarthy [25] gave a sketch of how his program would operate. He stated that a logical system more or less in the style of first order predicate calculus would be the right form for a declarative language.

First order logic has several attractive features for a declarative language. First it is well understood. Next, many people know it. It is clear that people can use it. Finally its mathematical structure has been well studied.

McCarthy envisaged a system in which first order predicate calculus statements would be read in, and some sort of deduction scheme (possibly modus ponens) would be used to deduce consequences until the answer came out.

Deduction is the central problem in using predicate calculus as a programming language. How could logical inference be set up on a computer? In 1964 several researchers attempted to build systems which used predicate calculus as an internal form. In Raphael's [29] SIR program, he allowed six predicate relations and wrote separate procedures for deduction

from each pair. Black [ 3 ] wrote a question answering system using several inference rules, each of which was a restricted form of modus ponens using a weak form of unification rule. Experience with these systems showed that they were hard to use and harder to extend. For example to add a new predicate to Raphael's program one had to be very familiar with the internals, and had to write a procedure specifying just how this predicate was related to all of the others. Cordell Green developed the QA1 question answering system [14] as another limited logic system. However, it could not do such simple problems as: given "Every person has two hands" and "John is a person", answer the question "How many hands does John have?"

## 1.3 Resolution

An alternative way to provide deduction with a computer system was clearly required. Researchers turned from ad hoc systems to uniform procedures suggested by mathematical logic. The search for such procedures dates back to Leibniz (1646-1716). The first effective procedure was found by Herbrand in 1930. It was, in fact, implemented on a computer by Gilmore [12]. This early work showed that more efficient techniques were necessary. Davis and Putnam [ 8 ] found several ways to improve Herbrand's procedure in 1960. Darlington [ 7 ] implemented their system in 1962. A major

breakthrough in theorem proving was the development of the resolution technique by Robinson in 1965 [30]. Resolution has two vital properties. First it is very easily implemented on a computer. Second, it is simple to work with mathematically.

## 1.4 Languages Based on Formal Logic

Resolution quickly captured the imagination of AI researchers. Cordell Green and Bertram Raphael developed the QA2 [15] system (1968) using resolution and first order logic. Coles [6] wrote an English-to-logic translator which would change subsets of English to the precise form of QA2 input. Early results using this system were very promising. Several very simple problems were solved, but difficulties began to appear. The speed of operation was very low. Small changes in formulation could make it impossible to solve problems. Researchers began to feel that some imperative code was necessary. Green [15] argued that one could conceal imperatives in the declarative formulation. He began to talk of "programming in the predicate calculus for efficiency", and suggested combining imperative and declarative code. He associated a LISP program with selected predicates and functions. The idea, which we will also use, was to execute the LISP program within the resolvent. For example (TIMES 2 2) could evaluate to 4. Using executable predicates,

Green developed QA3. This sytem could solve the 'Tower of Hanoi' problem, using an executable predicate to check legal moves. Executable predicates were effective, but executable functions were not, for the following reason: After evaluating the executable function, (TIMES 2 2) to 4, it could then be necessary to unify (TIMES 2 X) with 4. Since these cannot be made identical, functional evaluation seemed too difficult. One possible way of unifying (TIMES 2 X) with 4 consists of adding all the equality substitution rules and the axiom Y = (TIMES Z (QUOTIENT Y Z)); then it would be possible to apply substitution rules enough times to get

Z = (TIMES 2 X) ∨ ... P(X) ... ∨ ... Q(Z) ...

Resolution with the axiom would give

... P(QUOTIENT Y 2)) ∨ Q(Y) ...

Unification of Y with 4 and finally evaluation of (QUOTIENT 4 2) would then give 2. Unfortunately such an approach leads to an enormous number of resolvents. Since resolution is complete, partial evaluation is useful only when it reduces the number of clauses.

Other researchers questioned the division; "How much imperative vs. how much declarative?" Hewitt [21] developed PLANNER, a language which allowed intermixing fragments of LISP code and predicates within a modified resolution theorem prover. Here one could provide strategy functions which the built-in prover would call to

select clauses. Rulifson et al. developed QA4 [34], a language based on type theory (in order to get an ω-order logic) which allowed clauses to contain programs. These programs, written in a LISP-like notation, would act as a strategy. The difference between QA4 and PLANNER was that PLANNER had a built-in strategy which called user-supplied subroutines, while QA4 had the user program as the entire strategy. Several very impressive programs were written in PLANNER [44]. Unfortunately more and more code took on an imperative flavor. Sussman [41] argued that the built-in strategy of PLANNER was, in fact, the wrong one. In cooperation with McDermott he developed CONNIVER, a language in which users had complete control over the strategy being used. The whole discussion of strategies, often called procedural embedding, is still not ended. One alternative often suggested is that more powerful control structures should be added to the language. A series of languages, QLISP [33], SCHEME [42], etc., were developed to add control flow statements to a logical formalism. We will describe some of the details of these mechanisms in Section 3.4.

It is our feeling that much of this work has been predicated on the beliefs that general purpose uniform procedures had to be very slow, that resolution was developed to its utmost, and that, by their very nature, complete domain-independent strategies could not be made efficient. In our opinion such conclusions are premature. In the next chapter we will describe a generalization of resolution which will be more efficient than resolution alone. We will describe the Dilemma language, which is at about the declarative level of QA3, but is significantly more efficient.

## CHAPTER 2

## THE THEOREM PROVING SYSTEM

Theorems 1, 2, 4, 5 and 6 in this chapter have been proved elsehwere (Harrison and Rubin [18]); the proofs are repeated here for convenience and completeness.

### Resolution Proofs

At this point, we want to consider a difficult resolution problem. This problem, which in this formulation is near the limit which can be proved by current programs, consists in showing that $x \cdot y = (-x) \cdot (-y)$ in a ring. We will use a very simple formulation.

There is one binary predicate equality written as =, two binary functions +, and ·, and a unary function - (minus). We will write the clauses as implications for reasons which will become clear later. The input to the theorem prover is:

| | Clause | Function |
|---|---|---|
| 1. | $x + z = y + z \rightarrow x = y$ | unique sums |
| 2. | $x \cdot (y + z) = x \cdot y + x \cdot z$ | distributive law |
| 3. | $x \cdot y + z \cdot y = (x + z) \cdot y$ | |
| 4. | $0 = x + (-x)$ | additive inverse |
| 5. | $0 = -x + x$ | |
| 6. | $0 \cdot x = 0$ | a property of zero |
| 7. | $0 = x \cdot 0$ | |
| 8. | $x = z \wedge z = y \rightarrow y = z$ | transitivity of equality |
| 9. | $y = x \rightarrow x = y$ | symmetric property of equality |

| | | |
|---|---|---|
| 10. | $x = x$ | reflexive property |
| 11. | $w = y \wedge x \cdot w = z \rightarrow x \cdot y = z$ | substitution properties of equality |
| 12. | $w = x \wedge w \cdot y = z \rightarrow x \cdot y = z$ | |
| 13. | $w = x \wedge w + y = z \rightarrow x+y = z$ | |
| 14. | $w = y \wedge x + w = z \rightarrow x+y = z$ | |
| 15. | $a \cdot b \neq (-a) \cdot (-b)$ | negated theorem |

Assuming a resolution system which started with the negated theorem we could get any of a large number of essentially equivalent minimal refutations, each containing 13 resolutions. All of these refutations would involve 3 instances of clause 8, and 1 instance of each of clauses 1,2,3,4,5,6,7,9, 11,12,15. Such refutations would differ only in the order the refutations are done. For example the left distributive law could be used before or after the right.

## 2.2 Refutation Graphs

We would like to avoid regarding these as different proofs. To that end, we introduce the following definition.

We define a refutation graph G of a set S of unsatisfiable clauses as follows. Let R be a refutation of S. Consider an undirected graph G whose nodes are the clauses of S used in R (*if a clause of S occurs more than once in R, there should* be one node for each occurrence). Each branch in G is associated with two literals which resolve in R, and connects the clauses containing those literals (if merging or factoring occurs in R, there is a branch in G connecting every clause containing a

literal which merges to become the positive literal to every clause containing a literal which merges to become the negative literal). Figure 1 gives an example of a refutation graph. It is clear that a set S may have many refutation graphs, and that each refutation graph can be derived from one or more resolution refutations.

Performing a resolution on a refutation graph involves replacing the parent clauses by the resolvent, and applying the most general unifier to the other clauses of the graph. The literals of a resolvent will be considered to retain the association they had in the parent clauses. The definition of refutation graph used in this work is similar to but not identical with the resolution graph given in [45].

Figure 2 is a refutation graph of the set of clauses of the input problem. It is easy to see that this particular graph forms a tree, i.e. it has no cycles. Refutation trees are derived from resolution refutations which do not use merging or factoring (in fact from refutations which treat different occurrences of the same literal independently.

We will not consider the refutation graphs which are not trees until Chapter 6.

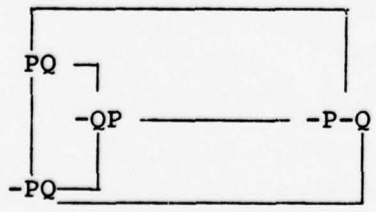



Figure 1

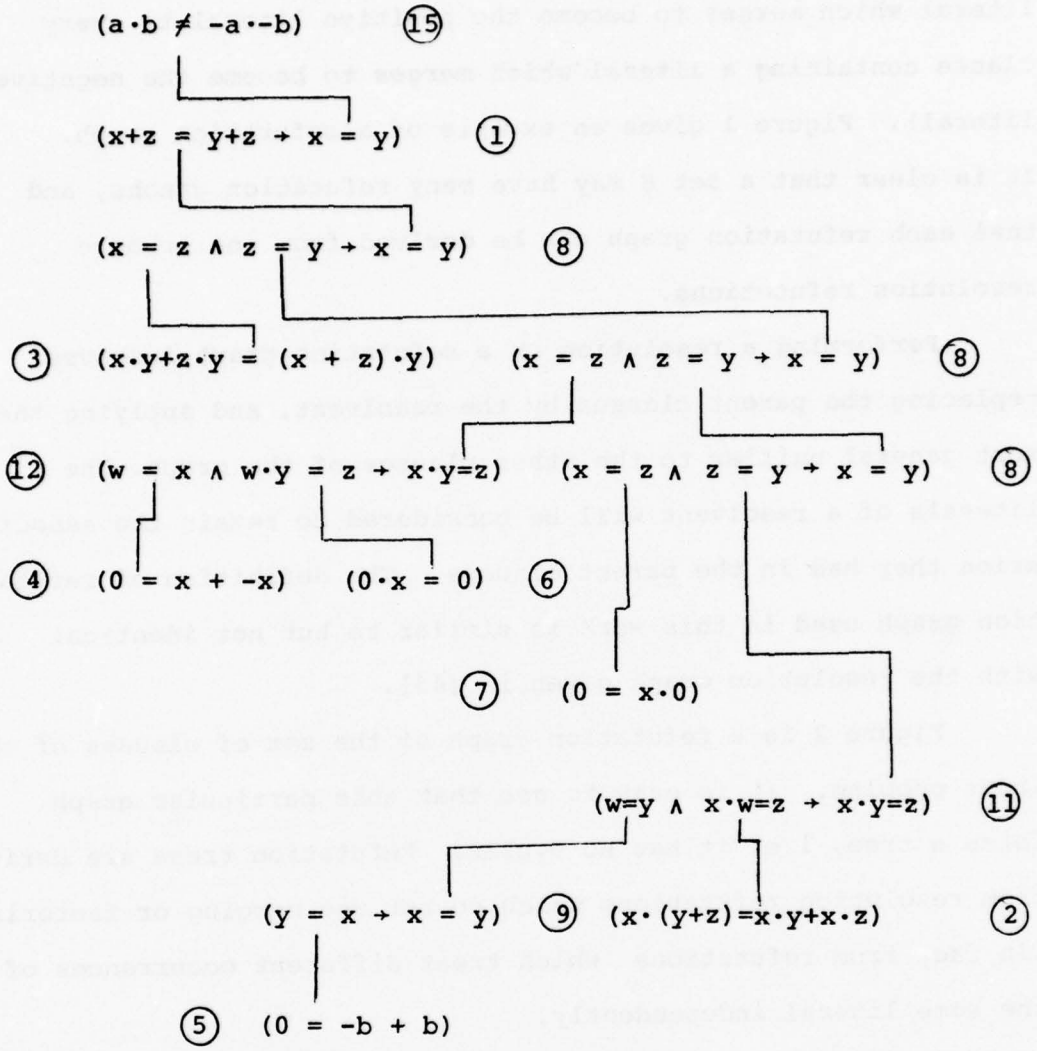



Figure 2. The Refutation Graph of the x·y = -x·-y Example.

If an unsatisfiable set of clauses has a refutation tree then it has a particularly easy sort of refutation.

Before we show this, we need to introduce the following definitions.

A simultaneous unifier (su) of a set of pairs $\langle x_i, y_i \rangle$, $i = 1,\ldots,n$, is a substitution $\lambda$ which satisfies $x_i\lambda = y_i\lambda$, $i = 1,\ldots,n$. If there exists an su for a set of pairs, the set is called simultaneously unifiable (which we also signify by su). We use mgu to stand for most general unifier. If the pairs $\langle x_i, y_i \rangle$ are su, then their most general simultaneous unifier (mgsu) is any $\lambda$ such that for any su $\mu$ there exists a $\sigma$ such that $\lambda\sigma = \mu$.

Lemma 1 (Andrews, 1968). The mgsu of a set of su pairs $\langle x_i, y_i \rangle$ is unique to within alphabetic variations, and is computable as the product $\lambda_1\lambda_2$, where $\lambda_1$ is the mgu of $\langle x_j, y_j \rangle$ and $\lambda_2$ is the mgsu of $\langle x_i, y_i \rangle\lambda_1$ for $i = 1,2,\ldots,j-1, j+1,\ldots,n$, and j can be chosen arbitrarily.

Proof: This was first proved, in a slightly more general form, in [ 2]. The proof is by induction on n. If n = 1 then the lemma is obvious.

We assume that there is a su called s of $\langle x_i, y_i \rangle$, $1 \leq i \leq n+1$, and a mgsu $\sigma$ of $\langle x_i, y_i \rangle$, $1 \leq i \leq n$. Let $L = \langle x_{n+1}, y_{n+1} \rangle$.

Since s is an su of the pairs up to n, there is a B such that $s = \sigma b$. Now $Ls = L\sigma b$ is unified, so b unifies $L\sigma$.

Since $L\sigma$ is unifiable there is a mgu $\tau$ of $L\sigma$. We assert that $\sigma\tau$ is the mgsu of the pairs up to n+1.

(1) $\sigma\tau$ is an su for $1 \leq i \leq n$, $\langle x_i, y_i \rangle \sigma$ is unified so $\langle x_i, y_i \rangle \sigma\tau$ is also. $L\sigma\tau$ is also unified by the definition of $\tau$.

(2) $\sigma\tau$ is an mgsu. Suppose $\gamma$ is an su of the pairs up to n+1; then $\gamma$ is an su of the pairs up to n. So $\gamma = \sigma u$. Since $L\gamma = L\sigma u$, u unifies $L\sigma$, thus $u = \tau c$. Now $\gamma = \sigma u = \sigma(\tau c) = (\sigma\tau)c$.

(3) Uniqueness. Assume $\tau, \sigma$ are two mgsu; then there exists d, e such that

$$\tau = \sigma d , \quad \sigma = \tau e$$

if P is any well formed formula then $P\tau$ and $P\sigma$ are alphabetic variants, so the unifier is unique to within alphabetic variants.

Now we may prove

Theorem 1 Given a refutation tree T for a set S of clauses, a merge-free and factor-free resolution refutation of S can be obtained from T by carrying out the resolutions corresponding to the branches of T in any order.

Proof: The refutation tree T contains a set of clauses each of which is in S, and determines the existence of an mgsu of the set of pairs of literals associated with the branches of T. Consider a particular order of the branches

of T, $B_1, B_2, \ldots, B_n$. We can perform the resolutions on the literals associated with the branches in this order. Since an mgsu of the pairs of literals exists, each of these resolutions can be performed. □

Now we will characterize what class of sets of clauses have a refutation tree.

Theorem 2 The following statements about a set S of clauses are equivalent:

(i) S has an input proof, using factoring and resolution.

(ii) S has a unit proof, using factoring and resolution.

(iii) S plus the factors of S have a refutation tree.

Proof: (i) and (ii) were proved equivalent by Chang [ 4 ], so we will just prove the equivalence of (i) and (iii). First, suppose S has an input proof I, using the clauses $I_0, I_1, I_2, \ldots, I_n$ in that order. Consider the refutation graph G of S corresponding to I. Process G as follows:

1. Set i = n.

2. If two literals of $I_i$ are associated with the same literal in another clauses, replace $I_i$ in G by the factor of $I_i$ which unifies these literals, making the corresponding modification in the branches of G and go to 2.

3. If a literal L in $I_i$ is associated with two other literals $L_1$ and $L_2$, do the following: Disconnect the branch associated with L and $L_1$; duplicate the graph containing $I_i$ neglecting all branches between $I_i$ and $I_j$ for $j < i$; reconnect this graph by adding a branch associated with $L_1$ and the copy

of L; and go to 3.

4. Set i = i-1; if i is nonzero, go to 2.

Each of the modifications in steps 2 and 3 does not destroy the fact that G is a refutation graph. Furthermore, this algorithm changes G to a tree, since in step 3 the graph containing $I_i$ is a tree. This can be seen by noting that just before step 4 each copy of $I_i$ is connected to exactly one $I_j$ for some $j < i$.

Now suppose S plus its factors has a refutation tree T. We can carry out the resolutions specified by the tree in any order, so we can construct an input proof by selecting an arbitrary clause of T first, and then performing the resolutions of T with input clauses (or factors of input clauses) in such a way that the literal resolved on is a member of the previous resolvent. □

Corollary 1 If S has an input (unit) refutation using factoring and resolution, then S plus its factors has an input (unit) resolution refutation without factoring.

Corollary 2 If S has an input refutation using resolution without factoring then S has a refutation tree.

## 2.3 Horn Clauses

Theorem 2 tells us when a refutation tree exists, but in a rather useless way. We first have to know if a set of clauses has a refutation of a certain form, so we have to give

a proof before we know if there is a refutation tree. We will now give syntactic conditions which will be sufficient to guarantee the existence of a refutation tree.

If we have a clause in disjunctive normal form

$$-L_1 \vee -L_2 \vee L_3$$

which has no more than one positive literal, we may write it as an implication.

$$L_1 \wedge L_2 \rightarrow L_3$$

Each such implication is a conjunction of zero or more atoms, followed by an implication sign, followed by either zero or one atom. A clause in this form is called a Horn clause. A set representing a conjunction of Horn clauses is called a Horn set. Henschen and Wos [19] showed:

If S is an unsatisfiable Horn set, then S has an input refutation without factoring.

Theorem 3. If S is an unsatisfiable Horn set, S has a refutation tree.

Proof: From the second corollary to Theorem 2.

Given a set of clauses in Horn form, a possible algorithm for finding refutation trees consists of:

(1) Form a pool of subtrees initially containing only negative clauses

(2) Select some subtree in the pool, call it T;

(3) Remove T from the pool.

(4) Pick some literal in T which has no associated branch.

(5) Compute all possible subtrees which link the selected literal of T to positive literals of input clauses.

(6) If any of these trees has no literal without an associated branch, otherwise add each of these trees to the pool and go to step (2).

It is easy to see that this algorithm is exponential in the number of branches of the final tree. In the next section we will show how to cut down the number of branches and thus reduce the complexity of this algorithm.

## 2.4 The Revised Unification Problem

In Figure 3, we draw the outline of the $x \cdot y = (-x) \cdot (-y)$ refutation tree.

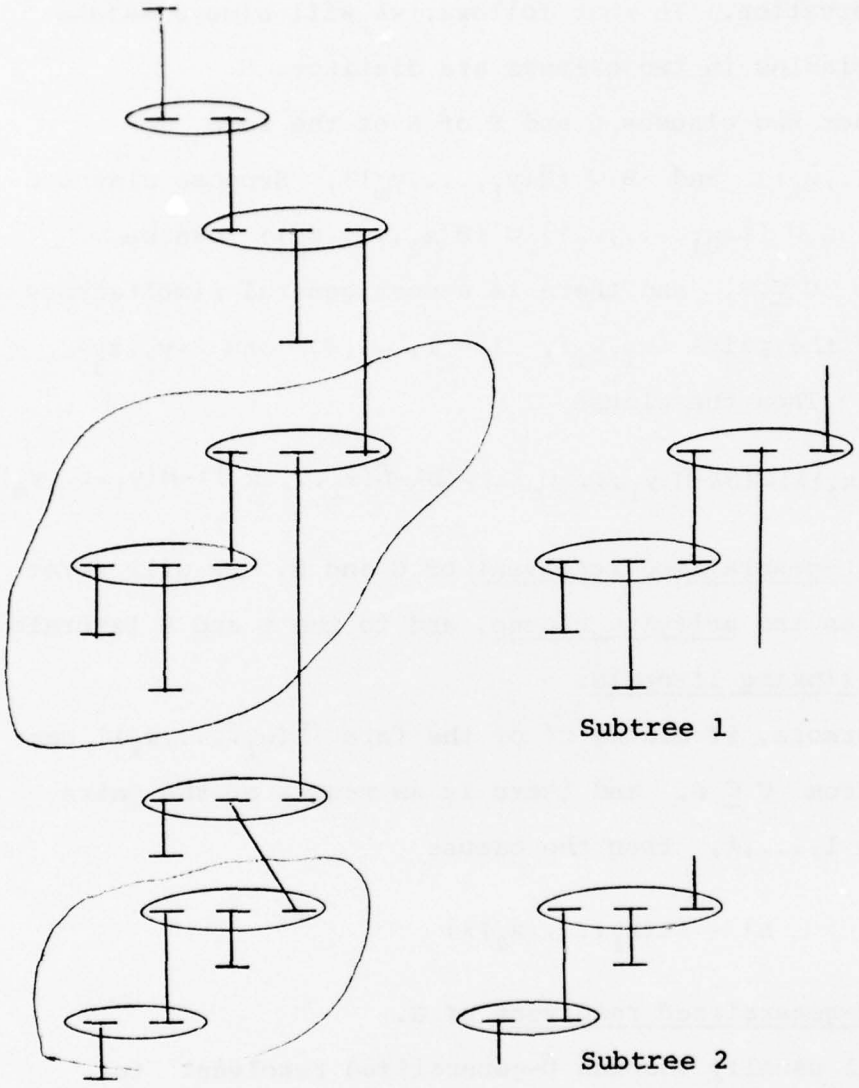



Figure 3. Outline of the Refutation Tree for $x \cdot y = (-x) \cdot (-y)$.

Here, we have selected several subtrees. If we regard the subtrees as units then the resultant tree has fewer branches. We may define a generalization of resolution based on this observation. In what follows, we will always assume that the variables in two clauses are distinct.

Consider two clauses G and H of S of the form $A \cup \{L(x_1,\ldots,x_\ell)\}$ and $B \cup \{\bar{M}(y_1,\ldots,y_m)\}$. Suppose clause C of the form $D \cup \{\bar{L}(w_1,\ldots,w_\ell)\} \cup \{M(z_1,\ldots,z_m)\}$ can be deduced from $U \subseteq S$, and there is a most general simultaneous unifier $\lambda$ of the pairs $\langle x_i,w_i\rangle$, $i = 1,\ldots,\ell$, and $\langle y_j,z_j\rangle$, $j = 1,\ldots,m$. Then the clause

$$(A\lambda - L(x_1,\ldots x_\ell)\lambda) \cup (B\lambda - \bar{M}(y_1,\ldots y_m)\lambda) \cup (D\lambda - \bar{L}(x_1,\ldots x_\ell)\lambda - M(y_1,\ldots y_m)\lambda)$$

is a <u>binary U-generalized resolvent</u> of G and H. We will refer to clause C as the <u>unifying clause</u>, and to the L and M literals of C as the <u>linking literals</u>.

Furthermore, if clause C' of the form $\{\bar{L}(w_1,\ldots,w_\ell)\}$ can be deduced from $U \subseteq S$, and there is an mgsu $\lambda$ of the pairs $\langle x_i,w_i\rangle$, $i = 1,\ldots,\ell$, then the clause

$$A\lambda - \{L(x_1,\ldots,x_\ell)\lambda\}$$

is a <u>unary U-generalized resolvent</u> of G.

We will usually shorten U-generalized resolvent to U-g-resolvent, and drop the words "unary" or "binary" when no distinction is being made.

We envisage U-g-resolution being used in the following way. Given a set S of clauses, select a subset U not including the theorem to be proved. Attempt to generate the null clause by repeatedly adding to S the U-g-resolvents of clauses in S-U. Thus the clauses in U are used essentially as rules of inference rather than clauses from which inferences can be made. Note that the proofs generated in this way should have fewer steps than standard resolution proofs, and these steps should be larger. The search strategy will involve the selection of (one or) two clauses from S-U, and the selection of (one or) two literals belonging to them which can be "linked" by a clause inferred from U.

The question then arises of the completeness and soundness of generalized resolution.

Theorem 4 U-g-resolution is sound.

Proof: Trivial.

Theorem 5 If S is a set of clauses with a refutation tree, then for any satisfiable $U \subset S$ S-U has a U-g-resolution refutation.

Proof: Consider a refutation tree T of S. We will show that the order of performing resolutions in T, which can be chosen arbitrarily, can be chosen so that it consists of a set of subsequences, each of which corresponds to a U-g-resolution step.

Either there is no clause in $U \cap T$, in which case S-U has a standard resolution refutation, or we can find a subgraph t of T satisfying:

(a) t has at least one node in U

(b) if a clause K is in $U \cap t$, all clauses connected in T to K are also in t

(c) t is connected

(d) all nodes of t not in U are connected in t to only one other node of t.

Thus t is a free tree with internal nodes in U.

Now since U is satisfiable, there must be at least one node of t in S-U. If there is exactly one, say K, then it must be possible to carry out all the resolutsion in t - {K}, leaving a single literal to resolve with a literal in K; in this case the result of carrying out all resolutions in t is a unary U-g-resolvent of K. Otherwise there are at least two nodes of t in S-U, say $K_1$ and $K_2$. $K_1$ and $K_2$ can be connected by a path P in t, and carrying out the resolutions along P is equivalent to forming a U-g-resolvent of $K_1$ and $K_2$. Note that in carrying out the U-g-resolution, more literals than $K_1$ and $K_2$ may be removed from T. However, this merely decomposes T into disconnected subtrees, one of which must be a refutation.

Thus in each case we can reduce T to T', a tree with strictly less branches than T, by carrying out resolutions corresponding to the formation of some U-g-resolvent R, with T' being a refutation tree for $S \cup \{R\}$.

Thus given any refutation tree T of S we can reduce it by sets of resolutions each corresponding to U-g-resolution steps until the refutation tree contains no more clauses of U. This reduced tree can be refuted by standard resolution steps not involving clauses of U, thus completing a U-g-resolution refutation of S-U. □

Corollary. Since in the proof above we do not require that identical literals be merged, the theorem remains true if we do only partial merging. In particular, if no two literals are ever identified.

In the case of binary U-g-resolution, we may view the generalization as a modification of unification.

Before: we had to find a substitution $\sigma$ which would make two formulas identical.

Now: We need to find a substitution $\sigma$ and a subtree T which make the formulas equivalent.

One way in which we may establish equivalence is to use a second, inner, theorem prover. Such an inner theorem prover provides three advantages:

(1) It needs to do simpler deductions

(2) The pair of literals selected by the outer theorem prover acts as a guide or higher level plan

(3) The outer theorem prover reasons in larger steps.

At best, we trade one proof of n steps for two each of about n/2 steps. Since the work is exponential in the number of steps, if we happen to be in the best case we get a

significant advantage. However, in the worst case, much computation will be done (repeatedly generating the same useless results) by the inner theorem prover, while the proof remains n steps long. In specific cases, it may be possible to eliminate large classes of potential unifying clauses. In the rest of this chapter, we will consider the important case in which U contains the axioms of equality.

## 2.5 Equality

Many schemes have been proposed to build a notion of equality into a theorem proving system. Robinson and Wos [31] noted that using equality as a binary predicate produced systems which were very slow. Experimentally, they found that introducing a three place predicate $P(x,y,z) \Leftrightarrow x \cdot y = z$ improved the performance of a resolution theorem prover by about 3 orders of magnitude.

Silbert [39] advanced the first special equality system. He had four rules of inference.

Rule 1 From $A \cup P(x_1,\ldots,x_n)$ and $B \cup \neg P(y_1,\ldots,y_n)$ where P is not equality, infer

$A \cup B \cup x_1 \neq y_1 \cup \ldots \cup x_n \neq y_n$ .

[generalized resolution]

Rule 2 From $A \cup x = y$, $B \cup u \neq v$ where A, B only contain equalities or inequalities y occurs only at the top level in A, $\sigma$ unifies y and v, infer $(A \cup B \cup u \neq x)\sigma$

[equality replacement]

Rule 3 From $A \cup F(x_1,\ldots,x_n) = y$ where A only contains equalities or inequalities, infer

$A \cup z_1 \neq x_1 \cup \ldots \cup z_n \neq x_n \cup F(z_1,\ldots,z_n) = y$

[equality substitution]

Rule 4 Delete literals of the form x = x.

The problem in Silbert's plan however where to apply rules 2 and 3. Using them everywhere allows all possible substitutions. His technique was never implemented.

The next proposed equality technique was paramodulation Suggested by Robinson and Wos in 1969 [31], it became the most often used equality method. In summary, it consists of the following.

Given two clauses

$$C$$

$$x=y \cup B$$

where

(1) t is a term in C

(2) $\sigma$ is the mgu of t and x ,

infer the clause

$$C_0 = (C \cup B)\sigma$$

Suppose that there are n occurrences of t, $\sigma$ in $C_0$; then we conclude the n clauses $C_1$ through $C_n$ where $C_i$ is the result of replacing the ith occurrence of $t\sigma$ in $C_0$ with $y\sigma$.

Robinson and Wos showed that "a ring is commutative if $\forall x, x^3 = e$ " could be proved in 49 steps of paramodulation or

136 resolutions. The proof of $x \cdot y = (-x) \cdot (-y)$ in the formulation above takes 11 paramodulation steps or 13 resolutions. Much work has been done with paramodulation, perhaps because it is easy to work with mathematically. Implementations have not shown clearly that it is efficient.

E-resolution, due to Anderson [1], uses both paramodulation and resolution. Two literals are selected; then using the unit equalities, all possible paramodulations of each literal are performed. Then all pairwise ways of unifying the literals are computed. An incomplete version was implemented [27].

Dixon [9] suggested Z-resolution. Basically, it consists of the following: to resolve two clauses $L_1 \vee A$ and $-L_2 \vee B$, first compute the sets $L_1$- VAR (of equality variants of $L_1$) and $L_2$-VAR (the set of equality variants of $L_2$). Next compute a set U of unifiers. Each $\sigma \in U$ will unify a member of $L_1$-VAR with some member of $L_2$-VAR. Finally, the Z-resolvents are computed by

$$(A \cup B)\sigma - \{L_1\sigma, -L_2\sigma\} \quad \text{for} \quad \forall\sigma \in U \ ;.$$

The equality variants of a literal $\ell$ are obtained as the resolvents of $\ell$ and clauses in a variant set V. V contains two literal clauses, where no variable occurs more than once in a single literal. Resolutions of elements of V produce tautologies. Z-resolution is a very powerful scheme and is perhaps the best of the equality methods. A particularly attractive alternative to Z-resolution is the following variation of our U-g-resolution technique.

## 2.6 Equality-Generalized Resolution

Consider two clauses G and H of the form $A \cup \{L(x_1,\ldots,x_\ell)\}$ and $B \cup \{\bar{L}(y_1,\ldots,y_\ell)\}$. Suppose clauses $D_i \cup \{w_i = z_i\}$ can be deduced from the set U of clauses such that either 1) there is a mgsu $\lambda$ of the pairs $\langle x_i,w_i\rangle$ and $\langle y_i,z_i\rangle$. Or, 2) if in addition L is the equality predicate, there is an msgu $\lambda$ of the pairs $\langle x_1,w_1\rangle$, $\langle z_1,y_2\rangle$, $\langle x_2,w_2\rangle$, $\langle z_2,y_1\rangle$. Then the clause

$$(G\lambda\text{-}L(x_1,\ldots x_\ell)\lambda)\cup(B\lambda\text{-}\bar{L}(x_1,\ldots x_\ell)\lambda)\cup(\cup_i(D_i\lambda\text{-}L(x_1,\ldots x_\ell)\lambda\text{-}\bar{L}(y_1\ldots y)\lambda))$$

is a U-equality-generalized resolvent of G and H. In what follows we will shorten U-equality-generalized resolvent to U-e-g-resolvent.

Note that for every U-e-g-resolvent there is a corresponding U-g-resolvent, but the reverse is not true.

In what follows we will refer frequently to the (infinite) set E containing the following equality axioms

$$\{x \neq y,\ y = x\}$$

$$\{x \neq y,\ y \neq z,\ x = z\}$$

and all axioms of the form

$$\{x \neq y,\ z \neq f(\ldots,x,\ldots),\ z = f(\ldots,y,\ldots)\}$$

$$\{x \neq y,\ \bar{p}(\ldots,x,\ldots),\ p(\ldots,y,\ldots)\}$$

for all function letters f and predicate letters p.

In practice we will always be concerned with the (finite) subset of E containing only function and predicate letters used in other clauses of a set S-E.

Theorem 6 If S is a set of clauses, including $\{x = x\}$, with a refutation tree, then there exists a linear U-e-g-resolvent refutation of S-U for any U satisfying

(i) $E \cap (S - U) = \emptyset$

(ii) U-E contains only positive unit equality clauses but not $\{x=x\}$.

Proof: The proof of this theorem is similar to that of Theorem 5. That is, any refutation tree T with at least one node in U has a subgraph t which is a free tree with internal nodes in U.

We can show that there is at least one pair of (terminal) nodes in $t \cap (S - U)$ which have a U-e-g-resolvent which can be obtained by carrying out only resolutions specified by t. For any internal node of t is in E, so is of one of the following forms:

$$\{x \neq y,\ y = x\}$$

$$\{x \neq y,\ y \neq z,\ x = z\}$$

$$\{z \neq f(\ldots,x,\ldots),\ x \neq y,\ z = f(\ldots,y,\ldots)\}$$

$$\{\bar{p}(\ldots,x,\ldots),\ x \neq y,\ p(\ldots,y,\ldots)\}$$

In each case the first literal has the same form as the last literal except for a substitution or, in the case of the

symmetry axiom, reversal of the arguments. This means that starting with any negative literal in t, we can find a sequence of resolutions in which all the resolved literals are identical except for substitutions (or argument reversals, if an equality literal).

Now we can choose this initial negative literal $\bar{L}(x_1,\ldots,x_\ell)$ to be in S - U. There must be at least one such literal, since not all positive literals in $t \cap U$ can be linked in t to negative literals in $t \cap U$ (this would give a path which would not end in a terminal node, contradicting the condition that t is a tree). The substitution path starting with $\bar{L}(x_1,\ldots,x_\ell)$, where L is not equality, must terminate in another clause in S - U containing a linking literal of the form $L(y_1,\ldots,y_\ell)$. Furthermore, the internal nodes on this path specify substitutions or reversals which indicate how $\bar{L}(x_1,\ldots,x_\ell)$ can be transformed into $L(y_1,\ldots,y_\ell)$. These internal nodes, together with some applications of the transitivity axiom if necessary, permit us to deduce from E clauses of the form

$$D_i \cup \{x_i = y_i\}$$

where the $D_i$ are sets of equality literals. Thus the application of the resolutions along such a path corresponds to an E-e-g-resolution step.

If, on the other hand, L is equality, the substitution/reversal sequence of internal nodes starting with $x_1 = x_2$ may end either in a node $\{y_1 = y_2\}$ in S - U, or in a positive

equality unit $\{y_1 = y_2\}$ in U. In both cases the set of internal nodes must enable us to define from U clauses of the form

$$D_1 \cup \{w_1 = z_1\}$$

$$D_2 \cup \{w_2 = z_2\}$$

with either $\langle x_1,w_1\rangle$, $\langle x_2,w_2\rangle$, $\langle y_1,z_1\rangle$, $\langle y_2,z_2\rangle$ simultaneously unifiable, or $\langle x_1,w_1\rangle$, $\langle z_1,y_2\rangle$, $\langle x_2,w_2\rangle$, $\langle z_2,y_1\rangle$ simultaneously unifiable. In the first case this sequence of resolutions is equivalent to a U-e-g-resolvent with linking literals $x_1 = x_2$ and $y_1 = y_2$. In the second case we can replace $y_1 = y_2$ in t with

$$\{y_1 \neq y_2,\ y_2 \neq y_2,\ y_1 = y_2\}$$

$$\{y_1 = y_2\}$$

$$\{y_2 = y_2\}$$

the last literal being an instance of x = x, which is in S-U. Thus the second case can be reduced to the first, so also corresponds to an U-e-g-resolution step. Thus given any refutation tree T of S, we can reduce that tree by applying sets of resolutions corresponding to U-e-g-resolution steps until the tree contains no more clauses of U. This reduced tree can be refuted by standard resolution steps not involving clauses of U, thus giving an U-e-g-resolution refutation. □

Corollary. As in Theorem 5, Theorem 6 remains true whether merging of literals is done or not.

In effect, Theorem 6 says that if in U-g-resolution U is constrained to be E, then we need only consider unifying clauses which have linking literals with the same predicate letter, and further, such unifying clauses have as their essential components other clauses which establish the equality of the arguments of the linking literals.

Theorem 6 thus gives us a generalization of the standard unification procedure. The standard procedure says that literals can be unified only if there are substitutions for variables which make the literals identical; the generalized procedure is to permit substitutions which enable the arguments of the literals to be proven equal. Theorem 6 shows that in the case of sets of clauses with a tree refutation the use of the equality axioms can be restricted to the problem of proving the arguments equal, and need not appear as clauses to be resolved upon.

Theorem 6 cannot be generalized to arbitrary sets of clauses; the E-unsatisfiable set of clauses:

$$\{\bar{P}(a),\ \bar{P}(b)\}$$

$$\{P(a),\ P(b)\}$$

$$a = b$$

cannot be proved unsatisfiable by the E-e-g-resolution procedure since the necessary factoring cannot be done. We conjecture, however, that if factoring and merging is generalized to use equivalence in an analogous fashion, then many of the completeness results of standard resolution procedures can be carried over.

Returning to the example, no proof of $x \cdot y = -x \cdot -y$ in the formulation given in Section 2.1, has been reported in the literature, though Wos et al. [31] proved it in a hybrid formulation (using a three-argument product predicate and also the equality predicate). Our formulation, being restricted to Horn clauses, must have an input proof starting with the negated theorem. The following is a standard input resolution proof of minimum depth (13 resolvents): Numbers to the right correspond to Section 2.1.

(R1) $a.b + x \neq -a.-b + x$ from (15) and (7)

(R2) $a \cdot b + x \neq y, \quad y \neq -a \cdot -b + x$ (R1) and (8)

(R3) $(a+b) \cdot b \neq -a \cdot -b + x \cdot b$ (R2) and (6)

(R4) $(a+b) \cdot b \neq y, \quad y \neq -a \cdot -b + x \cdot b$ (R3) and (8)

(R5) $w \neq a + x, \; w \cdot b \neq y, \; y \neq -a \cdot -b + x \cdot b$ (R4) and (11)

(R6) $0 \cdot b \neq y, \; y \neq -a \cdot -b + -a \cdot b$ (R5) and (3)

(R7) $0 \neq -a \cdot -b + -a \cdot b$ (R6) and (2)

(R8) $0 \neq x, \; x \neq -a \cdot -b + -a \cdot b$ (R7) and (8)

(R9) $x \cdot 0 \neq -a \cdot -b + -a \cdot b$ (R8) and (1)

(R10) $w = 0, \; x \cdot w \neq -a \cdot -b + -a \cdot b$ (R9) and (12)

(R11) $0 \neq w, \; x \cdot w \neq -a \cdot -b + -a \cdot b$ (R10) and (10)

(R12) $x \cdot (-y+y) \neq -a \cdot -b + -a \cdot b$ (R11) and (4)

(R13) $\square$ (R12) and (5)

The refutation tree corresponding to this proof is given in Figure 2. Note that axioms (9), (13), and (14) are not used, but that (10) would have been necessary in several other cases had other axioms (such as (1), (2), (5), and (6)) been specified the other way around.

A paramodulation proof would permit the elimination of the steps involving the substitution axioms (11) and (12). That is, (R6) could be obtained directly from (R4), and (R11) could be obtained directly from (R9). This gives an 11-step proof.

A proof using E-e-g-resolution can be much shorter, since all resolvents involving clauses in E are omitted. That is, only resolvents (R1), (R3), (R6), (R7), (R9), (R12) and (R13) appear explicitly, giving a 7-step proof. This proof arises in a number of ways: for example, an E-e-g-resolvent clause could be found from (R1) and (6), using one instances of (8); or (R1) and (2), using two instances of (8) and one of (11); or (R1) and (1) using three instances of (8); or (R1) and (5) using three instances of (8) and one of (12). In practice it would not be likely that the longer unifying clauses would be found; it would be more likely that (R1) would resolve with (5) using a simple substituion for the right-hand sides, and an instance of (8)

$$a\cdot b + -a\cdot z' \neq a,\ z \neq x'\cdot(y'+z'),\ a\cdot b + -a\cdot z' = x'\cdot(y'+z')$$

for the left-hand sides.

The S-e-g-resolution proof is even shorter than this, consisting of only 2 steps, (R1) and (R12). That is, (R1)

$$a\cdot b + x \neq -a\cdot\ -b + x$$

can be resolved directly with (5)

$$x'\cdot(y'+z') = x'\cdot y' + x'\cdot z'$$

The right-hand sides are unifiable immediately, using the substitutions $x' \leftarrow -a$, $y' \leftarrow -b$, $x \leftarrow -a \cdot z'$. Generalized unification of the left-hand sides requires that a substitution be found such that $(a \cdot b + -a \cdot z')\lambda$ can be proved equal to $(-a \cdot (-b+z'))\lambda$. This can be done, following the refutation tree precisely, as follows:

| | |
|---|---|
| $a \cdot b + -a \cdot b = (a + -a) \cdot b$ | from (6) |
| $a \cdot b + -a \cdot b = 0 \cdot b$ | from (3) and (11) |
| $a \cdot b + -a \cdot b = 0$ | from (2) and (8) |
| $a \cdot b + -a \cdot b = -a \cdot 0$ | from (1) and (8) |
| $a \cdot b + -a \cdot b = -a \cdot (-b+b)$ | from (4), (10) and (12) |

With the substitution $z' \leftarrow b$, this gives the null clause as the S-e-g-resolvent of (R1) and (5).

## 2.7 Coercion

To turn Theorem 6 into a useful procedure, we need a strategy for finding the proofs of equality.

Suppose we have two terms $t_1$ and $t_2$ which we would like to prove equal. Also suppose U contains a clause

$a = b \leftarrow D$ , where D is a conjunction.

We may break up the proof of equality into two subproblems. First to show that $t_1 = a$ and second, to show that $b = t_2$. We must decide which clauses of U to use on a given problem. We choose to use the following criteria.

Before starting to prove that $t_1 = a$ we impose the restriction that either

(1) $t_1$ or a is a variable or

(2) $t_1$ and a are the same constant or

(3) $t_1$ and a both start with the same function letter.

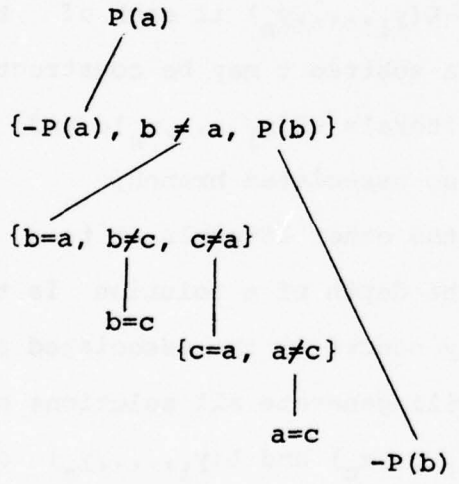
We then use the same criteria on $t_2$ and b.

Finally we place transitivity in both S and U.

The difference between coercion and U-e-g resolution may be seen in the following example.

| S - U | U |
|---|---|
| P(a) | a = c |
| -P(b) | b = c |
| x = x | E |

In U-e-g resolution we unify P(a) and -P(b) in the following tree.




Unification of P(a) and -P(b) produces the subproblem of proving b = a, which can be done within U. Thus U-e-g-resolution gives a one-step proof.

In coercion the transitivity axiom is in S rather than in U, so a coercion proof takes four steps. From an attempt to unify P(a) and -P(b), again we get the subproblem b=a. Now however we cannot apply transitivity inside unification, so the resolvent is

$$\{b \neq a\}$$

In the second step we resolve $b \neq a$ with transitivity to get

$$\{b \neq z, z \neq a\}$$

Two unary resolutions remove these literals. As this example shows, coercion consists in stopping the inner theorem prover just before applying transitivity. Transitivity is then applied by the outer theorem prover.

Finally unary resolutions complete the proof.

The coercion algorithm we use is defined as follows: An ordered pair $\langle\sigma,A\rangle$, where $\sigma$ is a set of substitutions and A is a set of literals, is called a solution of the coercion-unification of $L(x_1,\ldots,x_n)$ and $-L(y_1,\ldots,y_n)$ if each of the following conditions holds: (1) a subtree t may be constructed insode of unification; (2) the literals $-L(x_1,\ldots,x_n)\sigma$ and $L(y_1,\ldots,y_n)\sigma$ are in t and have no associated branch; (3) A is the set made up of all the other literals of t without associated branches. The depth of a solution is the number of times that transitivity occurs in the associated subtree.

The following algorithms will generate all solutions of the coercion unification of $L(x_1,\ldots,x_n)$ and $L(y_1,\ldots,y_n)$ of depth d or less.

To coercion-unify $L(x_1,\ldots,x_n)$ and $L(y_1,\ldots,y_n)$ to depth d,

(A) Solve the subproblem

$x_1=y_1$, $x_2=y_2$, ..., $x_n=y_n$ to depth d as follows:

(1) Let S = set of solutions of the subproblems $x_i = y_i$ to depth d as shown in (B) below

(2) If L is equality add to S the solutions of $x_1=y_2$, $x_2=y$ , as shown in (B) below.

(3) Return S.

(B) To solve a subproblem

$x_1=y_1$, $x_2=y_2$, ..., $x_n=y_n$

(1) let $S_1$ = set of solutions of $x_1=y_1$ to depth d as found by (C) below

(2) let $S_i$ = set of solutions of $(x_i=y_i)\sigma$ as found by (C) below where $\langle\sigma,A\rangle \in S_{i-1}$ for i = 2,...,n.

(3) Return $S_n$.

(C) To solve a subproblem

$x_i=y_i$ to depth d,

(1) let S = $\{\langle I, x_i=y_i\rangle\}$ where I is the identity substitution

(2) if $x_i$ and $y_i$ are not both functions, let $\sigma$ = most general unifier of $x_i$ and $y_i$ computed in the standard way; if $\sigma$ is defined add $\langle\sigma,\emptyset\rangle$ to S.

(3) If d = 0, go to (6).

(4) For each unit equality problem $\alpha = \beta$ form the two subproblems $x_i=\alpha$, $y_i=\beta$ with depth d-1; $x_i=\beta$, $y_i=\alpha$ with depth d-1.

(5) If the restrictions to Section 2.7 are met, add all solutions to these subproblems, found by (B) above to S.

(6) If $x_i$ and $y_i$ are functions starting with the same symbol add to S all solutions of coercion-unifying $x_i$ and $y_i$ to depth d.

(7) Return S.

For the unary case, we have:

To coercion-unify $x \neq y$

apply algorithm (C) above.

Finally, we prove below that we get the same deductions from either coercion of U-e-g-resolution. The advantage of coercion is that the outer theorem prover gets to see intermediate results and on that basis possibly suspend the computation of a unifier.

Theorem 7 If S is a set of clauses, including $\{x \neq y, y \neq z, x = z\}$, with a refutation tree, then there exists a linear coercion refutation of S-U for any U satisfying

(i) $E \cup (S - U) = \{x \neq y \vee y \neq z \vee x = z\}$

(ii) U-E contains only positive unit equalities but not $\{x=x\}$.

Proof: We will show that any U-e-g resolvent is produced by a series of coercion resolvents which do not involve any uses of transitivity inside of unification. We do this by induction on n, the number of uses of transitivity in the U-e-g resolution.

If n = 0, the U-e-g resolution and the coercion resolution are the same.

Assume the induction hypothesis is true for U-e-g resolutions with n or less transitivity applications. Consider a U-e-g

resolution with n+1 transitivity steps. We select one of these n+1 transitivity applications, say:

$$\{a \neq c,\ c \neq b,\ a = b\}$$

The $a = b$ must be linked to an $a \neq b$. If this $a \neq b$ is in a clause in S-U then we may resolve $a \neq b$ and transitivity directly. Otherwise this $a \neq b$ is in U, and therefore in E, since U-E contains only positive unit equalities. There are two cases: either it is on a chain of the type described in Theorem 6, or it is not. In the first case it can be removed by a U-e-g-resolution, and in the second must be produced by some U-e-g resolution as one of the resultant literals. In both cases this U-e-g-resolution will involve no more than U applications of transitivity, and so can be done by coercion.

Finally the two literals $a \neq c$, and $c \neq b$ may each be removed by a unary coercion. Such a deduction is allowed since the subtrees linked to these literals in the U-e-g resolution each contains no more than n applications of transitivity.

We have shown that no applications of transitivity are required inside unification; since using transitivity inside of unification can only increase the number of solutions, we may apply it selectively without removing completeness. Therefore in part (C) step (5) any set of restrictions may be used.

## 2.8 Comparison of Coercion and Resolution

The results reported below were obtained from a straightforward Horn clause resolution program which was modified to implement coercion. A switch could be set in this algorithm so that it would revert to the standard unification algorithm for comparison purposes.

Horn clause resolution was chosen for its simplicity. We use input resolution, with the literals in a clause maintained in the original order, and resolution permitted only on the first literal. We prefer input resolution, since at any point in an input resolution procedure, there are a number of resolvents which can be worked on (the "pool") and the order in which they are chosen does not affect the possible resolutions which can be done; this would appear to be a significant advantage when trying to compare procedures which are highly dependent on heuristics.

Only three heuristics were actually used. Resolvents with complexity (in effect the number of symbols in the expression) greater than some cut-off point were discarded; resolvents were selected in order of increasing complexity and increasing depth if complexity was equal; coercion was restricted to a maximum depth of three. Resolvents which were alphabetic variants of previously encountered clauses were also discarded.

Example 1 is the ring theory problem discussed in Section 2 above. The generalized unification procedure found the proof discussed, using axioms (1) through (8) only. The results are summarized in Table I. The standard resolution procedure ran out of time after having generated 30% more resolvents but only getting half of the proof.

Example 2 is a somewhat simpler ring theory problem. That is, from the axioms

(1) $x = 0 + x$

(2) $x \cdot (y+z) = x \cdot y + x \cdot z$

(3) $x + z = y + z \rightarrow x = y$

together with appropriate equality axioms, to prove

(4) $x \cdot 0 = 0$

In standard resolution, this proof could be obtained in the following 11 steps:

| | | |
|---|---|---|
| (R0) | $a \cdot 0 \neq 0$ | negated theorem |
| (R1) | $a \cdot 0 + z \neq 0 + z$ | (R0) and (3) |
| (R2) | $0 + z \neq a \cdot 0 + z$ | (R1) and symmetry |
| (R3) | $x \neq y,\ 0 + y \neq a \cdot 0 + x$ | (R2) and substitutivity |
| (R4) | $0 + (x \cdot y + x \cdot z) \neq a \cdot 0 + x(y+z)$ | (R3) and (2) |
| (R5) | $a \cdot 0 + x \cdot (y+z) \neq 0 + (x \cdot y + x \cdot z)$ | (R4) and symmetry |
| (R6) | $v \neq x \cdot (y+z),\ a \cdot 0 + v \neq 0 + (x \cdot y + x \cdot z)$ | (R5) and substitutivity |
| (R7) | $x \cdot (y+z) \neq v,\ a \cdot 0 + v \neq 0 + (x \cdot y + x \cdot z)$ | (R6) and symmetry |
| (R8) | $w \neq y+z,\ x \cdot w \neq v,$ $a \cdot 0 + v \neq 0 + (x \cdot y + x \cdot z)$ | (R7) and substitutivity |

(R9) $x \cdot w \neq v,\ a \cdot 0 + v \neq 0 + (x \cdot 0 + x \cdot w)$ (R8) and (1)

(R10) $a \cdot 0 + x \cdot w \neq 0 + (x \cdot 0 + x \cdot w)$ (R9) and

(R11) □ (R10) and (1)

Our standard resolution program actually ran out of time after generating 754 clauses, getting (R4) but not (R5). There were 429 clauses left in the pool at this point. Note that this proof is made longer than it might be by the use of the particular forms of the axioms, necessitating several applications of the symmetry axiom.

The generalized unification proof took just the following two steps:

(R0) $a \cdot 0 \neq 0$

(R1) $a \cdot 0 + x \neq 0 + x$ (R0) and (3)

(R2) □ (R1) and (1) and (4)

This proof appeared as the $17^{th}$ clause generated, at which time there were 4 clauses in the pool. The step from (R1) to (R2) is by no means obvious, and proceeds as follows: First (R1) is resolved against (1), with the unification subproblem

$$a \cdot 0 + x = x$$

which is unified with (2) giving the subproblem

$$a \cdot 0 + x \cdot (0 + z) = x \cdot 0 + x \cdot z$$

which generates the subproblem

$$0 + z = z$$

which unifies with (1) reversed. The results are summarized in Table II.

| Coercion | | Resolvent | Standard Resolution | |
|---|---|---|---|---|
| Resolvent Number | Pool Size | | Resolvent Number | Pool Size |
| 2 | 0 | (R1) | 2 | 0 |
| 13 | 4 | (R2) | 40 | 24 |
| 21 | 3 | (R3) | 156 | 79 |
| 65 | 14 | (R4) | 452 | 214 |
| 128 | 21 | (R7) | >779 * | 351 * |
| 136 | 26 | (R8) | | |
| 370 | 30 | (R9) | | |
| 597 | 46 | □ | | |
| | | | * out of time | |
| 80 resolvents accepted | | | 130 resolvents accepted | |
| 236 seconds | | | >512 seconds | |

Table I. Comparison of Proofs of $a \cdot b = -a \cdot -b$

| Coercion | | Resolvent | Standard Resolution | |
|---|---|---|---|---|
| Resolvent Number | Pool Size | | Resolvent Number | Pool Size |
| 1 | 0 | (R1) | 4 | 15 |
| not needed | | (R2) | 35 | 20 |
| | | (R3) | 57 | 35 |
| | | (R4) | 338 | 208 |
| | | (R5) | >754 * | 429 * |
| | | ... | | |
| 17 | 4 | □ | | |
| 12.7 seconds | | | *out of time | |
| | | | >512 | |

Table II. Comparison of Proofs of $x \cdot 0 = 0$

## CHAPTER 3

## THE DILEMMA PROGRAMMING LANGUAGE

While it would be possible to base a computer language solely on the coercion mechanism such a system would then be deficient in several aspects. First, since it would not have a procedural component, though we could state any fact or concept, it would be hard to tell the theorem prover exactly when to use these facts or concepts. We believe that such additional information is often best given in imperative form.

A second deficiency occurs when there are many extraneous ground clauses. If, for example, we wished to have the entire multiplication table of the hardware described by a set of clauses, the storage and retrieval problems would be terribly complex. In any given run only a few of these would be relevant. A far simpler technique would be to generate ground clauses as needed.

The third deficiency is the duplication that arises when there are clauses which are equivalent. For example in a geometry program we could have a clause containing the line AB and a second clause containing the line BA. Ideally we would like to represent clauses in a canonical form deleting duplicates.

The final deficiency that we wish to consider is that duplicate computations are often necessary. If in the course of proving two lines equal we prove that a pair of triangles is congruent, we would like to record that fact. Proofs of congruence should only be done once. Furthermore, once we know that two triangles are congruent, there are

a number of immediate consequences; for example, corresponding angles are equal. Often it is computationally advantageous to draw these consequences immediately.

In this chapter we will describe the pure language and a way to remove some of these deficiencies by introducing imperatives. We next will give the syntax and semantics of Dilemma using coercion and imperatives. Chapters 4 and 5 discuss a sample program which proves geometry theorems. Finally, Appendix I contains an interpreter of Dilemma written in BALM.

## 3.1 Horn Clauses as Procedures

Logical statements of the form:

$$\neg P_1 \vee \neg P_2 \vee \ldots \vee \neg P_n \vee Q$$

$$Q$$

$$\neg P_1 \vee \neg P_2 \vee \ldots \vee \neg P_n$$

are said to be in Horn form. Each such statement has at most one positive literal. Many useful structures may be described by a collection of Horn statements. The following result due to Horn [22] characterizes such structures. We first define the notion of "closed under direct product". Suppose S is a set of statements, with $m_1$ and $m_2$ two models for S. If $m_1 \times m_2$ is also a model of S we say S is closed under direct product. Horn showed that if a set of clauses was closed under direct product then there exists an equivalent set of clauses which is in Horn form. For example since we know that if $G_1$ and $G_2$ are two groups, then $G_1 \times G_2$ is also a group, there is a set of logical statements in Horn clause form which will describe groups.

Statements of the form $\neg P_1 \vee \neg P_2 \vee \ldots \vee \neg P_n \vee Q$ may be viewed as implications $P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow Q$ and as procedures with name $Q$ and body $P_1$ through $P_n$. We interpret such a procedure to mean: $P_1$ through $P_n$ are sufficient conditions to establish Q.

Statements of the form Q may also be viewed as implications $\rightarrow Q$ or as procedures with name Q, and null body. We interpret such a procedure to mean Q is an immediate result.

Finally statements of the form $\neg(P_1 \vee \ldots \vee P_n)$ are also implications

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow \text{contradiction}$$

We interpret this as initial goals, i.e. report success if it is possible to achieve all of $P_1$ through $P_n$.

When there are arguments involved we coerce them together. For example:

Given the procedures:

(1) $P(b,a) \rightarrow$ contradiction

(2) $Q(x,a) \rightarrow P(a,x)$

and the unit equality $a = b$ we execute the body of (1) by first proving that $b = a$ inside of unification and then binding x to a. The body of (2) becomes $Q(a,a)$.

The term coercion is based on the coercions of ALGOL 68.

We note that bindings in a call affect the caller as well as the called program. We could, for example, have the assignment statement:

$$\rightarrow SETQ(x,x)$$

which could be called by either SETQ(x,z) or SETQ(z,x).

The basic execution semantics of our language consists of:

(1) let N = set of negative clauses

(2) pick some c in N; remove it from N

(3) pick some literal $\ell$ in c

(4) using coercion, compute the set S of generalized resolvents of subroutines with C, resolving on $\ell$ and the name of the subroutine

(5) if $\square \in S$, stop; otherwise, go to (2).

As we have said, this form of language would be very inefficient. The next two sections will describe additional features which improve the performance.

Since the basis of the language will be a two level theorem prover applied to Horn clauses, we have chosen to call the language Dilemma. Before we give a formal statement of Dilemma, we will give an example.

3.2 An Example (The Jekyll and Hyde Problem)

Suppose we wish to tell the machine of a kindly old doctor named Jekyll. Jekyll is the name of an individual or a constant in our logical universe. To tell DILEMMA about him we write:

```
CONSTANT JEKYLL END;
FACT1 ASSERT DOCTOR(JEKYLL):
FACT2 ASSERT KINDLY(JEKYLL);
F3 ASSERT OLD(JEKYLL);
F4 ASSERT HUMAN(JEKYLL);
```

(we have underlined keywords in these examples; the underlining is not part of the actual computer input.) Each of the statements generates a positive unit clause. Now we tell DILEMMA the nature of humans.

```
F5 THEOREM FALLIBLE(X) FROM HUMAN(X);
```

F5 is an implication: for all x, if x is human then x is fallible. To ask the question "is anyone fallible?" we may write:

```
PROVE (FALLIBLE(X));
```

PROVE statements generate negative clauses. In this case NOT FALLIBLE(X). This clause unifies with F5 to give the resolvent -- NOT HUMAN(X). In turn this resolves with F4 to give the resolvent □. Thus we have proved a fallible individual exists, namely Jekyll.

A slightly more complex question is "is there a fallible doctor?".

PROVE(DOCTOR (X) ∧ FALLIBLE(X));

The literal NOT DOCTOR(X) unifies with FACT1 to give the resolvent -- NOT FALLIBLE (JEKYLL). This generates NOT HUMAN(JEKYLL) via F5. As above a resolution with F4 produces the null clause. If there were a few other doctors in our universe of discourse, say Strangelove and Schweitzer we could add the inputs:

```
CONSTANT STRANGELOVE, SCHWEITZER END;
F6 ASSERT DOCTOR(STRANGELOVE);
F7 ASSERT DOCTOR(SCHWEITZER);
```

Asking the same question:

```
PROVE(DOCTOR(X)∧FALLIBLE(X));
```

gives the following resolvents:

```
NOT FALLIBLE(STRANGELOVE)     from F6;
NOT FALLIBLE(SCHWEITZER)      from F7;
NOT FALLIBLE(JEKYLL)          from FACT1.
```

Each of these is executed in parallel with F5 to give:

```
NOT HUMAN(STRANGELOVE)
NOT HUMAN(SCHWEITZER)
NOT HUMAN(JEKYLL)
```

The last of these yields the null clause via F4. The X in the PROVE statement is assigned the value JEKYLL since that is the only solution which produces the null clause. The other assignments of X are "undone" as those parallel paths of execution terminate unsuccessfully.

Suppose we add still another individual to this system, the diabolical Mr. Hyde.

```
CONSTANT HYDE END;
F8 ASSERT DIABOLIC(HYDE);
```

Up to this point DILEMMA has been similar to older AI languages. Now we come to a basic difference. We will say that Jekyll and Hyde are the same. One way to do this would be to write:

```
S1 ASSERT SAME(HYDE,JEKYLL);
```

Of course we must define the effects of SAME.

```
S2 THEOREM DOCTOR(X)      FROM DOCTOR(Y)    ∧ SAME(X,Y);
S3 THEOREM KINDLY(X)      FROM KINDLY(Y)    ∧ SAME(X,Y);
S4 THEOREM OLD(X)         FROM OLD(Y)       ∧ SAME(X,Y);
S5 THEOREM HUMAN(X)       FROM HUMAN(Y)     ∧ SAME(X,Y);
S6 THEOREM FALLIBLE(X)    FROM FALLIBLE(Y)  ∧ SAME(X,Y);
S7 THEOREM SAME(X,Y)      FROM SAME(Y,X);
S8 THEOREM SAME(X,Y)      FROM SAME(X,Z)    ∧ SAME(Z,Y);
S9 ASSERT  SAME(X,X);
SA THEOREM DIABOLIC(X)    FROM DIABOLIC(Y)  ∧ SAME(X,Y);
```

If we ask "is there a diabolic doctor?"

```
PROVE (DOCTOR(X)∧DIABOLIC(X));
```

the resolvents are:

1. NOT(DOCTOR(Y) ∧ SAME(X,Y) ∧ DIABOLIC(X))
2. NOT(DIABOLIC(JEKYLL)
3. NOT(DIABOLIC(STRANGELOVE))
4. NOT(DIABOLIC(SCHWEITZER))

In stage two the resolvents are:

1, 5. NOT(DOCTOR(Z), SAME(Y,Z), SAME(X,Y), DIABOLIC(X))

6. NOT(SAME(X, JEKYLL), DIABOLIC(X))

7. NOT(SAME(X,STRANGELOVE), DIABOLIC(X))

8. NOT(SAME(X,SCHWEITZER), DIABOLIC(X))

2, 9. NOT(DIABOLIC(Y), SAME(JEKYLL,Y))

3,10. NOT(DIABOLIC(Y), SAME(STRANGELOVE,Y))

4,11. NOT(DIABOLIC(Y), SAME(SCHWEITZER,Y))

In stage three the resolvents include:

NOT(DIABOLIC(HYDE))
NOT(SAME(JEKYLL,X), DIABOLIC(X))
NOT(SAME(X,Z), SAME(Z,JEKYLL), DIABOLIC(X))
NOT(DIABOLIC(JEKYLL)

Finally in stage 4 we produce the null clause.

Another way available in DILEMMA to describe the intimate relationship between Jekyll and Hyde is the coercion

H1 COER HYDE,JEKYLL;

H1 tells DILEMMA that the constant HYDE may be "COERced" into the constant JEKYLL when necessary for unification.

In standard unification we seek to find a substitution which will make two literals identical. In DILEMMA we require only that the literals become equivalent. Coercions are used to specify facts to an inner theorem prover. This prover, invoked during unification, attempts to prove the equivalence.

So, using H1 rather than the S's we have:

PROVE(DOCTOR(X) ∧ DIABOLIC(X));

1. NOT(DIABOLIC(JEKYLL))
2. NOT(DIABOLIC(SCHWEITZER))
3. NOT(DIABOLIC(STRANGELOVE))

In stage 2 we get the null clause by resolving 1 with F8 using H1.

The Dilemma system has been developed as an extension of the BALM4 programming language [16]. All of the BALM4 features are available to the Dilemma programmer. Several additional data types and structures are provided. We will assume that readers are familiar with the BALM4 language and will freely make use of the concepts and terminology defined in [17].

We will use the following notation:

| | |
|---|---|
| <e1>,<e2> | will be two expressions |
| <item> | will be any valid BALM4 item |
| <lit> | will be a literal |
| [x]* | will be 0 or more occurrences of x |
| <pos lit> | will be a literal which is interpreted to be positive. |
| <name> | will be any BALM identifier |
| <proc> | will be a BALM identifier |

## 3.3 The Syntax of a Basic Dilemma Program

### 3.3.0 Constants

Constants are defined by the CONSTANT statement in the form:

CONSTANT item,item,...,item END;

The items may be any expression. Usually, however, they are numbers or BALM4 identifiers. One CONSTANT statement adds to

the next. So if we use:

```
CONSTANT A,B,C,D END;
CONSTANT 1,2,RTANG,3,4 END;
```

we would have nine constants.

In input form constants are just the items from CONSTANT statements. However, internally (and in output) an extra set of parentheses are added to turn the constant into a function of zero arguments, so the input constant RTANG becomes (RTANG) internally.

### 3.3.1 Expressions and Literals

Expressions and literals occur in two forms in Dilemma. The first of these in called input form. Here, Skolem functions follow the rules for BALM4 expressions; e.g.

F(X), G(X,Y), S(X, F(Y)) .

Any BALM4 identifier (which has not been in a CONSTANT statement) inside of an expression is a variable and is local to the Dilemma statement containing that expression. Function and predicate symbols are just BALM identifiers. In the same way literals (in input form) follow the rules for BALM4 expressions; e.g., P(F(X)), P(X), Q(F(X), Y). Each time a predicate or function is used it must have the same number of arguments.

The other form called internal form is more LISP-like: all commas are removed, the parenthese are shifted to

include the predicate or function symbol, and variables are changed to numbers; e.g.

P(F(X), X, F(X)) is changed to (P (F1) 1 (F 1))

Internal form is used for output.

### 3.3.2 Unit Positive Clauses

The simplest Dilemma statement is the assertion:

<name> ASSERT <literal>;

For example:

L1 ASSERT EQLINE (L(X,Y), L(Y,X));

Assertions are positive unit clauses; they need not be ground.

### 3.3.3 Implications

In Dilemma an implication may be written:

<name> THEOREM <pos literal> FROM <literal> [∧<literal>]*;

The idea is that the conjunction of literals is sufficient to prove the <pos literal>. An example:

DESCARTES THEOREM EXISTS(X) FROM THINKS(X);

### 3.3.4 Negative Clauses

A negative clause is always the last input. It may be specified in one of the following ways:

(1) PROVE(<lit>); The lit is a unit negative clause

(2) PROVE(<lit>[∧<LIT>]*); There is one negative clause. Each literal must be proved.

(3) PROVE(LIST(<lit>[∧<lit>]*,...,<lit>[∧<lit>]*));
Prove any one of the conjunctions.

The third form is an "or of ands". The clauses are ordered. It is more efficient to enter clauses in increasing complexity.

The name of an assertion or implication is required. It is used to label output and internally to reference the clause. (Internally, the corresponding clause is actually assigned as the value of the identifier used to name it.)

### 3.3.5 Coercions

Coercions are used to state that two expressions are equal (or equivalent). If <e1> and <e2> are predicates then we will talk about their equivalence. Otherwise these are expressions and we talk of their equality. In either case we write <e1> = <e2>. To say that <e1> = <e2>, we write:

<name> COER <e1>, <e2>;

Often we may find a condition under which <e1> = <e2>. The general form is

name COER <e1>, <e2>, <lit>[∧<lit>]*;

Suppose H(X) implies F(X) = G(X). Then we may write:

HFG COER F(X),G(X),H(X);

If two ground items are not equal this may be specified by

DIFFERENT(<item1>,<item2>)

### 3.3.6 CPROCS

In order to have a useful language we need to be able to encode lots of tricks and heuristics. Dilemma provides a number of methods which let the user alter the normal flow of control.

Normally a coercion is used whenever applicable. However, we frequently know the restrictions on when to apply a particular coercion. Suppose we have a coercion C COER X,Y and a BALM procedure P ; the statement C CPROC P is of the

general form

<cname> CPROC <proc>;

which associates the procedure and the coercion.
We call P the CPROC of the coercion.

A CPROC is called just before the associated coercion is applied. Each CPROC has three arguments:

1. item from negative clause to be unified
2. item from mixed or positive clause to be unified
3. set of solutions already found (NIL if none is found)

The solutions are each a pair: a list of added literals, followed by a binding list.

The arguemtns to a CPROC are in internal form (the LISPish way to write a clause without commas). CPROCs return either:

1. NIL -- don't apply the coercion; or
2. TRUE -- use the coercion; or
3. a number -- which becomes an additional upper bound on depth of coercions used. Thus if a CPROC returns 0 then the coercion will be applied but no coercions will be used below it.

CPROCs often will use the function GROUND(X) which is true if x is a ground expression, CONSTANT(X) which is true if x is a constant, and the global variable UNIFDEPTH which gives the current recursive level inside unification. UNIFDEPTH = 0 at the top.

Some typical CPROCs are:

```
*       cnly use the coercion if no other solutions have been found
        UNIQUE = PROG(N,P,SOL), SOL EQ NIL END;
*       only use the coercion at the top level arguments
        TOP = PROC(N,P,SOL), UNIFDEPTH EQ 0 END;
*       only use the coercion if one of the two expressions is ground
        ONEG = PROC(N,P,SOL), GROUND(N) OR GROUND(P) END;

        A rather odd way to define a CPROC is:
        C CPROC PROC(N,P,SOL), SOL EQ NIL END;
```

Here we have defined the CPROC and associated the coercion in one statement.

A number of global variables affect just how coercions will be applied.

| | |
|---|---|
| BVARS | if true don't apply any coercions whenever both terms are variables. Normally this is true. |
| ONEWAY | if true apply the coercions in one direction. That is, if C COER L,R is a coercion only try to use L with the item from the negative clause and R with the item from the positive clause. Normally this is false. |
| MAXCOERS | the search for a unifier uses this value as a depth bound. 0 would prevent any coercions from being used. The switch can be used to turn DILEMMA into a standard input resolution theorem prover. |

The actual algorithm used to apply coercions is given below:

To unify N and P with the coercion C COER L,R; :

(1) If N and P are variables, and BVARS is true, bind N to P and return

(2) If POSSIBLE(N,L) and POSSIBLE(P,R) then

a) if there is a CPROC associated with C execute it; return false if it returns false

b) otherwise, if the current depth is less than MAXCOERS compute the set of unifiers which simultaneously satisfies UNIFY(N,L) and UNIFY(P,R).

(3) If ONEWAY is true, then return; otherwise, apply (2) to the coercion COER R,L; and add any new solutions found.

In the above, POSSIBLE(X,Y) is true if X is a variable or Y is a variable or X and Y start with the same function symbol.

## 3.4 PARTIAL EVALUATION

Sometimes we know exactly how to solve a problem or some part of a problem. For instance we could have a particular interpretation which suggests values of functions. Suppose we have a predicate

$$P(x,y,z) \leftrightarrow x \cdot y = z$$

If x, y, and z are all integers, we may check this predicate in a model, the computer hardware arithmetic. The check is done by the following sequence: "multiply x times y and compare with z". This sequence is an imperative explaining how to check items.

If, on the other hand, x were a variable we would like to execute the following: if y and z are integers and $y \neq 0$ then

$$set\ x = z \cdot y^{-1}$$

In addition, once a resolvent is computed, there are often simplifications to be made. For example, we could represent items in a canonical form. Or, we could evaluate functions, e.g. S(S(0)) goes to 2. At the same time we may add heuristic functions.

Partial evaluation is a technique of associating functions written in a standard computer language with predicates and Skolem functions. The collection of lower level functions is called the computational base. Whenever we know how to achieve some part of a goal in a simple way we may add a computational

function using all the standard tricks of the computer programmer.

We may associate with each N place function N+1 BALM procedures. Each time we bind a variable which occurs inside a function we will execute the associated procedures. The first N of these procedures corresponds to the arguments of the function. They are executed only if the argument is a variable. The output of each procedure will be a new binding for the variable. The last procedure is associated with the function as a whole. It will return a set of bindings and a value which is to replace the function. Each of these procedures gets the function as its argument. A typical function would be the successor function S(n). It is mapped to n+1 if n is an integer, and otherwise it returns the symbolic quantity S(n).

In a similar way, we can associate a BALM procedure with each predicate. The result of executing such a procedure should be either a replacement literal, TRUE indicating that the literal may be assumed true or FALSE indicating that the literal cannot be proved. These procedures may also bind variables.

Several previous attempts have been made to use partial evaluation. Applied only to ground clauses and then only allowed to return a truth value, ad hoc partial evaluations could be constructed easily. However, when the attempt was made to generalize to some variant of the scheme we have

suggested, a problem appeared. Suppose we have a successor function S(N) where for instance S(S(0)) evaluated to 2. The difficulty occurs when we have to unify 2 and S(N). Since no binding will make 2 and S(N) identical, they could not be unified. Yet for completeness we certainly need them to be. The coercion technique provides a way of retaining these unifications. All we need do is find conditions under which 2 and S(N) are equivalent. We could for example use the coercion:

SUC COER X, S(N), EQ(X, PLUS(N,1)) .

A second use of partial evaluation is to adjust the global search strategy. In the course of executing a procedure we may change a number of global variables. These delay the expansion of certain nodes or cause others to be processed immediately. The idea is that low level computations done inside of partial evaluation can be used to direct the search strategy. If, for example, we had a predicate INT(x,y,z) to stand for "integrating x with respect to y gives z" we could have a clause of the form

INT(PLUS(P,Q),Y,PLUS(R,S)) ← INT(P,Y,R) ∧ INT(Q,Y,S)

If we wanted to use this rule only when P was simpler than PLUS(P,Q) we could add an additional predicate:

INT(PLUS(P,Q),Y,PLUS(R,S)) ← INT(P,Y,R) ∧ INT(Q,Y,S)
∧ SIMPLEFORM(P,PLUS(P,Q))

SIMPLEFORM could reject terms of a complex nature by returning FALSE if P or PLUS(P,Q) were too complex.

By embedding heuristics within the partial evaluation scheme our system functions rather differently than other AI languages. In a language like PLANNER, a clause is represented by an ordered sequence of pairs. Each pair consists of a literal and a filter.

Prior to resolving on a given literal, the filter is called to produce an ordered sequence of clauses to resolve with. Often the filter acts as a generator producing the elements of this sequence one at a time. Resolution is then attempted only between the literal and the first clause in the sequence. An automatic backup system allows the program to eventually compute the other resolvents. In our system, a clause consists of a set of literals and strategies. When we have selected a literal, we resolve it with all other possible clauses. We put all strategies which were in either parent clause into the resolvent. Finally we execute all of these strategies. Besides setting global variables, the strategies may return TRUE, meaning delete this strategy from the resolvent, FALSE, meaning delete the resolvent, or another strategy to be put into the resolvent. If a strategy returned a pointer to itself, the strategy would be invoked again for descendants of the resolvent.

The PLANNER scheme has the advantage that if the filter is reasonable, it can cut down the number of attempted resolutions. On the other hand, we see a number of advantages to our system. First, since the strategy obtains the value of

variables after the resolvent has been computed it may make a more informed decision. Second, all the strategies are evaluated at each point rather than just one. Third, since strategies may remain in the descendants, a strategy may wait in making a decision until additional information is available. Finally there are many more possibilities for parallelism since we compute all the resolvents at once, and then evaluate the strategies simultaneously. Perhaps we should add one more difference: suppose a literal $\ell$ could be advantageously resolved with a literal in a clause C. In PLANNER the recognition of that advantage would be computed by a filter associated with the literal $\ell$. In our language, the recognition occurs by a strategy associated with C. If we add a new clause, we need only put the strategy in that clause. In PLANNER we would haveto change the filters associated with all old literals which could resolve with this clause. Thus, in Dilemma, as we add new clauses, telling of better ways to do things, less changes need to be made in the original clauses.

### 3.4.1 PARTEVAL

The PARTEVAL statement is used to associate BALM procedures with the partial evaluation of function and predicate symbols. These procedures which we will call PE's are executed just after the resolvent is computed.

<pred symbol> PARTEVAL <proc>;

<func symbol> PARTEVAL <proc>;

The argument of a PE is the part of the literal which starts at the predicate or function symbol. It is almost in internal form. However, variables are changed to the BALM id's

X1,X2,X3,...,X12. If we have

F PARTEVAL FPROC;

P PARTEVAL PPROC;

and the resolvent contains the literal (P (F 1) (A)) then the execution operates as follows. First FPROC will be called with argument (F X1). If FPROC returns its argument, PPROC will be called with (P (F X1) (A)) as its argument. If a PE is used for a function, it must return a replacement expression (one possibility is its argument). To reject a clause the function need only set the global variable RETAIN to FALSE.

PE's used for predicates may return

1. TRUE -- delete the literal
2. FALSE -- delete the clause
3. replacement literal.

For PE's which are to be associated with predicates an alternative diction is available:

<pred> PART <i> = <proc>; $1 \leq i \leq 6$

In this form the PROC is called only if the $i^{th}$ argument of the literal is a variable. It may return:

1. TRUE -- delete the literal
2. FALSE -- delete the clause
3. A binding for the variable which will make the literal true
4. A replacement literal.

PE procedures generally consist of a few calls to built-in functions. We will first describe several of these functions in detail and then later give a complete list.

The built-in function BIND(expl,exp2) is used to assign values to Dilemma variables: expl is replaced by exp2. One of the odd properties of BIND is that anything may be bound to anything else. If we bind (F X1) to (G X1) then all (F X1) expressions in the clause are changed to (G X1). Also if we bind a variable other than X1,X2,...,X12 to a value that fact is recorded on a special list which is associated with the clause. PE's may use BIND to communicate. The function ISBOUND(ID) finds out if ID is bound or not. The result of ISBOUND is TRUE or FALSE. The global variable BNDVAL gets the bound value. CRACK is a generalized assignment, e.g. CRACK(pattern,expression). Its first argument is a pattern. This is matched with the second argument; structure and constants must match. Whenever a variable is matched with a subexpression, an assignment is made. An additional Dilemma statement allows variables to be typed:

TYPE <id> = <proc>;

Typed variables are global. Prior to CRACK assigning a value to a typed variable, the procedure is called with the value as its argument. The assignment is carriedout only if the procedure returns true. CRACK itself returns a value: TRUE, if the pattern matches the expression and all assignments could be carried out; and FALSE otherwise. Suppose ARG = '(P (1) (2) (3)). Then CRACK(' (T X Y Z), ARG) would be true and would set the values of T to 'P, X to '(1), Y to '(2), and Z to '(3).

If we had said TYPE X = INTQ; TYPE Y = INTQ; TYPE Z=INTQ; the assignments would be T = 'P, X = 1, Y = 2, and Z = 3. If ARG2 = '(P X1 3 4) then CRACK (' (T X Y Z), ARG2) would be false.

Suppose we want to do addition by means of a PE. We could have a predicate ADDSUP(X,Y,SUM) which is true if X + Y = SUM. If X, Y and SUM are all numbers, say 1, 2, 5, then a PE associated with ADDSUP would be called with (ADDSUP (1) (2) (5)) as its argument.

We could type the arguments as above. Then CRACK would set X = 1, Y = 2, Z = 5. So the addition PE would be

```
ADDSUP = PROC(PRED),
   IF NOT(CRACK('(T X Y Z), PRED)) THEN PRED
   ELSE (X + Y) EQ Z END;
```

## 3.5 Completions

Suppose we had a negative clause $\neg(P(x) \wedge Q(x))$ and resolve away the $P(x)$ in a series of steps. If x receives the final value $f(g(a))$, we may summarize the closed subtree by the positive literal $P(f(g(a)))$.

We will call such a positive literal the completion of the associated negative literal. In a sense, the completion of a literal is a lemma of the proof. The lemma procedure of model elimination [ 4 ] finds lemmas in this way. Since each completion is a new unit clause implied by the set of clauses, we may add the completions to the original set. Furthermore, since these units have already been generated by the proof, resolutions involving these units are likely to be useful. To that end, when we produce a completion we enter a unit resolution mode attempting to produce new clauses specialized to the problem. For example, suppose we try to prove two lines equal by showing that corresponding triangles are congruent. When we remove the literal for convergence we may resolve the completion with clauses of the form

CONGRUENCE → EQLINE

CONGRUENCE → EQARG

to derive a set of units saying that corresponding points of congruent triangles are equal. Of course, completions are able to generate a lot of additional clauses. In the implementation a number of processes are available to cut down

completions. These processes allow us to associate BALM procedures with predicate symbols. Whenever a literal is completed the BALM procedure associated with its predicate symbol is called. This procedure may return:

(1) TRUE -- add the completion to the input clauses

(2) FALSE -- don't add it

(3) A list of names of other clauses -- the system will do unit resolutions with these clauses and any of their descendants produced by this or earlier completions.

### 3.5.1. Completion Syntax

The statement

<proc> COMPLETES <pred>;

causes procedure <proc> to be called when a positive unit with the predicate symbol <pred> is produced. The procedure receives the unit as the argument. The procedure may return

(1) TRUE -- add the unit to the positive clauses

(2) FALSE -- don't add the unit

(3) A list of names of theorems. This is interpreted to mean: do one step of unit resolution with the unit on each of these theorems and on any dependents of the theorems produced by this or earlier completions. (An empty list of names acts like FALSE).

Effectively completions add a unit resolution component. When completions are provided each clause may contain literals of the form (MARK x), where x is a literal previously resolved upon.

## 3.6. How to Control the Search

Clauses are selected in complexity order (lowest merit first). The procedure COMPLEX(x) computes the complexity of the list of literals x.

The function SELECT1(x) returns one of the literals in the list of literals x. This will be the next literal resolved on.

The built-in functions COMPLEXITY and SELECT1 are defined as

COMPLEXITY(x) ↔ number of symbols in x

SELECT1(x) ↔ return the hd of x.

A number of functions are provided which may be used when writing new versions of these procedures. (Many of these functions are similar to standard BALM functions but use EQUAL rather than EQ.)

| | |
|---|---|
| DELETE(item,list) | returns list with item removed. changes list in place. |
| VARIABLE(item) | returns true if item is a variable e.g. an atom which is not a constant |
| LOOKUP(item,list) | returns the value associated with item on list. False if item is not on list. |
| MEMBER(item,list) | returns true if item is on list |
| SETPROP(id,prop-name,value) | sets the property prop-name of variable id to value. |
| POSLIT OF name | returns a copy of the positive literal of a clause name. |

Also three global variables are of interest when writing these functions.

| | |
|---|---|
| NEGS | all contexts not yet resolved on |
| DONE | all contexts already used |
| NOTEQLIST | ground pairs known not to be unifiable. |

## 3.7 Built-In Equality

The clause

REFLEX ASSERT EQ1(x,x);

is built-in. The clause

THEOREM EQ1(x,y) FROM EQ2(x,z) ∧ EQ2(z,y);

is built-in. EQ1 and EQ2 are unifiable (however a coercion must be used in the unification).

## 3.8 System Parameters

Users may wish to change the following control variables:

| | |
|---|---|
| CLAUSEPRINT | print each resolvent if true |
| NEGPRINT | print each negative clause before finding possible resolvents |
| MAXLITS | maximum length of a clause |
| MAXLEVEL | maximum depth of a clause (0 is the depth of an input clause) |
| MAXCOERS | maximum number of coercions used in 1 unification |
| MAXCPX | maximum complexity value |

| | |
|---|---|
| MAXNEST | maximum number parentheses of a clause |
| DEBUG | If true the system will print intermediate results; this is for internal debugging of the system rather than of DILEMMA programs. However, it often will prove useful in removing some of the trickier bugs. |

## 3.9 A Complete Example -- Missionaries and Cannibals Using PE Functions

As an example, we will give a program for the usual missionary and cannibal problem. Suppose there are 3 missionaries and 3 cannibals on one side of a river, and a boat which holds at most 2 people. We wish to find a sequence of moves which gets everyone across the river without ever having a group of missionaries outnumbered.

We will use the following predicates.

RIGHT(c,m) ↔ the problem may be solved with C cannibals, M missionaries and the boat on the right

INRANGE(X) ↔ X is less than or equal to the number of people who can be on the boat

LTE1(x,y) ↔ $x \leq y$

OK(C,M) ↔ it is OK to have C cannibals and M missionaries together

OKBOAT(X) ↔ $0 < x \leq 2$

LEFT(C,M) ↔ equivalent of right with C cannibals,
M missionaries, and the bota at the left.

BEST(X,Y) ↔ a heuristic predicate

EQL(x,y) ↔ x = y

and the following functions:

PLUS(x,y) ↔ x + y

MINUS(x,y)↔ x - y

The DILEMMA program for solving this problem is as follows:

```
CONSTANT 0,1,2,3,4,5,6 END;
RT  THEOREM RIGHT(CX,MY) FROM
INRANGE(X) ∧ INRANGE(Y) ∧ LTE1(X,CX) ∧ LTE1(Y,MY)
∧ OK(PLUS(MINUS(3,CX), X), PLUS(MINUS(3, MY), Y))
∧ OK(MINUS(CX,X), MINUS(MY,Y))
∧ OKBOAT(PLUS(X,Y)) ∧ LEFT(PLUS(MINUS(3,CX),X),PLUS(MINUS(3,MY),Y))
∧ BEST(PLUS(MINUS(3,CX),X), PLUS(MINUS(3,MY),Y));
LT  THEOREM LEFT(CX,MY) FROM
INRANGE(X) ∧ INRANGE(Y) ∧ LTE1(X,CX) ∧ LTE1(Y,MY)
∧ OK(PLUS(MINUS(3,CX),X),PLUS(MINUS(3,MY),Y))
∧ OK(MINUS(CX,X), MINUS(MY,Y))
∧ OKBOAT(PLUS(X,Y)) ∧ RIGHT(PLUS(MINUS(3,CX)X),PLUS(MINUS(3,MY),Y))
∧ BEST(MINUS(3,CX), MINUS(3,MY));
INZERO THEOREM INRANGE(X) FROM EQL(X,0);
INONE  THEOREM INRANGE(X) FROM EQL(X,1);
INTWO  THEOREM INRANGE(X) FROM EQL(X,2);
EQL    ASSERT  EQL(X,X);
```

```
TYPE XV = INTQ;  TYPE YV = INTQ;
LTE1 PARTEVAL PROC(PRED);
  IF NOT(CRACK(=(T XV YV),PRED)) THEN PRED ELSE XV LE YV END;
MINUS PARTEVAL PROC(X FN),
  IF NOT(CRACK(=(T XV YV),FN)) THEN FN ELSE XV-YV END;
PLUS = PROC(FN),
    IF NOT(CRACK(=(T XV YV),FN)) THEN FN ELSE XV+YV END;
PLUS PARTEVAL PLUS;

OKA THEOREM  OK(C,M) FROM LTE1(C,M);

OKB ASSERT   OK(C,0);

OKBOAT THEOREM OKBOAT(X) FROM LT1(X,3) ∧ LT1(0,X);

LT1 PARTEVAL PROC(FN), IF NOT(CRACK(=(T XV YV),FN)) THEN
    FN ELSE XV LT YV END;
* BEST ADJUSTS THE HEURISTIC VARIABLE MERIT
BEST PARTEVAL PROC(PRED), BEGIN( ),
    IF NOT(CRACK(=(T XV YV),PRED) THEN RETURN PRED,
    MERIT = -(XV + YV), RETURN TRUE
    END END;
* THE PROVE STATEMENT
PROVE(LEFT(3,3));
```

The actual run of this example produces a 93 level proof generating some 678 resolvents.

## 3.10 Implementation Outline

The information in this section may be useful for writing fancy PEs and CPROCs. Internally, a clause is represented by an 8 element vector called a context, whose elements are:

(1) id *0xx is a generated name

(2) pointer to negative parent

(3) pointer to literals (as a tree structre)

(4) length in negative literals

(5) mixed or positive parent

(6) special bindings

(7) depth of clause (0 is an input clause)

(8) merit (actually complexity) of clause

The variables in a clause are represented by numbers. Constants, function symbols, and predicates are heads of lists. Variables are not.

### 3.10.1 Macros

The following set of macros is provided for accessing parts of a clause:

(1) NAMEOF(context) - name of the clause

(2) NEGPAR(context) - returns vector pointer to parent

(3) LITLIST(context) - literals in internal form

(4) CLENGTH(context) - length in terms of number of literals

(5) MIXEDPAR(context) - returns vector pointer to parent

(6) BINDING(context) - special bindings

(7) DEPTHVAL(context) - depth of clause

(8) MERITVAL(context) - merit of clause

The above set can be used as the left-hand side of an assignment statement.

### 3.10.2 Functions

BIND(expl,exp2) - replace expl by exp2 in the clause; retain this if expl is a variable other than an x

ISBOUND(var) - see if this resolvent or an earlier parent had an assignment for this variable (returns true or false) (sets NBDVAL=binding value if found)

ISBOUND(var,c) - similar to ISBOUND(var) but starts search at context c

GROUND(x) - true iff x is a ground item

INPUTCL(context) - true iff context is an input clause

### 3.10.3 Other Details

Three global variables are often used by the PE programmer.

CURCONTEXT - current resolvent in packed vector form

DEPTH - depth of curcontext; level 0 = input clause

MERIT - merit of curcontext; "lowest merit first" is the selection rule.

A listing of the system, together with more detailed information on the implementation, is given in Appendix I.

## 3.11 Dilemma Syntax Summary

### Basic Objects

| | |
|---|---|
| names | BALM objects of type identifier |
| literals | |
| lists of literals | one or more literals separated by ∧ |
| procs | BALM objects of type code |
| predicate symbols | BALM objects of type identifier |
| function symbols | BALM objects of type identifier |
| constants | functions of zero arguments, or (identifier, in input form) |
| contexts | vector forms of clauses (used as a internal form). |

### Statement Types

| | | |
|---|---|---|
| (1) | DIFFERENT(x,y) | indicates that the ground items x and y are not unifiable |
| (2) | CONSTANT x,y,z END; | x,y,z will each be constants and are BALM identifiers |
| (3) | name ASSERT literal | defines a positive unit clause |
| (4) | name THEOREM literal FROM list of literals | defines a mixed clause. The literal is positive, each of the list of literals is negative. |
| (5) | name COER expl,exp2; | simple coercion. expl is equal (or equivalent to) exp2. |

(6) name COER expl,epx2, list of literals
compound coercion, expl equivalent or equal to exp2 under conditions (list of literals)

(7) PROVE(list of literals); last statement in a program. The list of literals is taken to be a negative clause; may be written as PROVE(LIST(list of lits, list of lits, ..., list of lits)); if there is more than one initial negative clause.

(8) proc COMPLETES predicate; associates the procedure with the predicate for completions

(9) proc CPROC name; The procedure <proc> will be executed during unification before the COERcion <name>.

(10) pred PART i = proc; pred will be executed after computing the resolvent, provided the ith argument of the predicate is still a variable

(11) pred PARTEVAL proc; proc will be executed just after the resolvent has been computed.

(12) TYPE var = proc; proc will be executed just before an assignment is made to the var by CRACK.

## Chapter 4

## The Geometry Program

We have claimed that the Dilemma programming language is extremely expressive, and that complex processes may be encoded in relatively simple ways. In order to illustrate the use of Dilemma and to give some evidence in support of this claim, we will describe a program which produces proofs for a small subset of elementary geometry theorems. The subset includes the equality of angles and of the length of lines, the parallelism of lines, and the congruence of triangles.

Gelernter [10] wrote the first program which could produce proofs for geometry. He introduced the notion of reasoning backwards. Here, one starts with the theorem, replaces it by the conditions which imply it, and then, in turn, replaces those conditions by their antecedents. This process is continued until the conditions have been replaced by axioms. Backward reasoning, which has also been called "Goal Directed Reasoning", is always used by Dilemma programs except when a completion is being applied.

Gelernter also introduced three major heuristics. Each of these involved the use of a semantic model. The program was given a "picture" in the form of coordinates of each point. In the first heuristic, goals would be rejected if they were false in this picture. Additionally, if the goal

contained a free variable, then the program would replace the goal by the set of instances using only the constants in the picture. False instances would immediately be rejected. This pruning heuristic trimmed the search tree from 1000 alternatives at each stage to five. Our program will allow free variables to remain in goals; since we don't need to pick particular instances, each clause will represent a number of different alternatives. The rejection heuristic would be successful only if every alternative could be rejected, at once. It will turn out that we do not need to use an analytic geometry model to reject goals.

The second use of a model was to evaluate the truth of "self-evident" statements. Such statements include, for example, facts about the ordering properties of points on a line, or the intersection properties of lines on a plane. If the statement was true in the model, it was assumed "proved by observation". In the Dilemma program, these "self-evident" truths are computed by partial evaluation. A third use of the model was to identify a given angle or line with all of its names, and to assume equality between any two different names of the same object. We use coercions to allow the alternative names of lines and angles to be equivalent.

The second major heuristic introduced by Gelernter was a point construction routine. When all else failed, the program would pick two points in the picture and draw the line between them. This line would then be extended to all of its inter-

sections with the other lines of the picture. Each point of intersection would become a new constant. Finally the program would start again. Our program will do constructions by unification, a way which we believe more natural.

Syntactic symmetry was the last of Gelernter's heuristics. Here, one associated a function $\sigma$ with each model. Two goals would be considered variants of each other if $\sigma$ applied to the first gave the second. The program would need to prove only one variant of each goal. The Dilemma notion of alphabetic variant is certainly much weaker. It is not clear just how often additional clauses could be removed if syntactic symmetry had been included.

Gelernter's program was quite long; about 20000 lines of code. The examples G1 through G5, which we will describe below were all solved by it. Several researchers attempted to clarify how these heuristics worked. It has been felt that the model heuristics were highly efficient, that they could be used in areas other than geometry, and that they were necessary to backward reasoning.

Goldstein [13] wrote a program for geometry theorem proving based on Gelernter's. Goldstein's system called BTP (Basic Theorem Prover) again was goal directed. He tried to develop a high-level, "natural" formalism to represent geometry knowledge as programs. He introduced the idea of canonical naming. Great simplifications would result if lines could have unique images. His system worked as follows:

Points could be ordered by their position in a diagram. Lines would be denoted by storing their end points in increasing order. Angle ABC would imply that A was less than C. Goldstein's program was also based on an analytic geometry model. It was written in a somewhat modified MICRO-PLANNER. Here, the simple automatic backup mechanisms proved to be an obstacle. First it was necessary to program explicit checks for duplicate goals. Second, automatic backup requires that any search be bounded. Thus his program could not create new points. Finally it was necessary to give up completely on one goal before trying an alternative. The problems G0 through G4 were solved by the BTP program.

Goldstein then proposed a second program to be called PTP (the Plausible-move-generating Theorem Prover). This program, which was not actually implemented, would do constructions in the following way: Whenever a clause involving a new point occurred, a plausible construction procedure would be used to find a new point meeting the conditions specified by the clause. Goldstein's idea was that a special procedure could be written for each type of construction and each clause that introduced a new point. Unification would select the correct clause, and thereby one or more useful move generators.

Nevins [28] tried to clarify the use of a model by writing an additional geometry program. He argued that the essential information held in a model was not numeric; in fact,

that it was the canonical ordering, the equivalence of names, and information on what looked parallel. His program ordered points lexicographically. No diagram was required. Nevins felt that the use of a model had forced previous researchers to write goal-directed programs. In his system, he included both forward and backward reasoning. The heart of his technique was a separately written LISP procedure for each paradigm of geometry. This program, which is perhaps the most efficient geometry theorem prover, has no provision for generating new points. The only aspects of a model which are used are nonnumeric. There is a list of lines which look parallel. Our own system is highly goal-directed, and is designed to construct new points through unification.

## 4.1 Basic Objects and Their Intended Meanings

We have 11 constants. RTANG and STANG represent constant angles. RTANG is 90 degrees, STANG is 180°. We have nine constants intended to represent points: E, P, K, N, M, C, D, Q, and B. We will introduce a total ordering on these constants by E < P < K < N < M < C < D < Q < B. This particular arrangement is produced by the BALM IFROMID primitive.

Only three logical functions are required. The first, L(X,Y) is intended to map pairs of points to lines. The second A(X,Y,Z) is intended to map triples of points to angles. Using partial evaluation we will associate canonical ordering procedures with these functions. The third Skolem function, MIDPT(X,Y) is intended to mean the point midway between X and Y.

The predicates allowed in this system are:

(1) CONG(X,Y,Z,U,V,W) ↔ triangle XYZ is congruent to triangle UVW

(2) EQLINE(S,T) ↔ the length of line S is equal to that of line T

(3) EQANG(X,Y) ↔ the size of angle X equals that of angle Y

(4) PARALLGM(X,Y,Y,Z) ↔ X,Y,U,V are the corners of a parallelogram

(5) COLIN(X,Y,Z) ↔ points X,Y,Z are collinear in that order.

(6) PARALLEL(S,T) ↔ line S is parallel to line T

(7) DOWN(X,Y,Z,U,V,W) ↔ a heuristic predicate defined below

(8) DOWN30(X) ↔ a heuristic predicate defined below

(9) DIFF3(X,Y,Z) ↔ X, Y, and Z are distinct points.

## 4.2 The Logical Axioms

We will now describe the clauses containing geometric information. This is the information given to the outer theorem prover.

Segment Equality

(1) a line is always equal in length to itself.

LREF ASSERT EQLINE(X,X);

(2) Two segments are equal if they both equal a third.

LTRANS THEOREM EQLINE(X,Y) FROM EQLINE(X,Z) ∧ EQLINE(Z,Y) ∧ DOWN30(X);

The introduction of a new line through transitivity is generally a poor way to do a proof. The predicate DOWN30 will make a clause 30 points more complex. This is one of the ways in which a predicate may be used to effect the search strategy.

(3) Sums of equal segments are equal.

LPART THEOREM EQLINE(L(X,Y), L(U,V))
FROM EQLINE(L(X,R), L(U,S)) ∧ EQLINE(L(R,Y), L(S,R))
∧ COLIN(X,R,Y) ∧ COLLN(U,S,R);

This axioms corresponds to the following picture:

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A HIERARCHICAL TECHNIQUE FOR MECHANICAL THEOREM PROVING AND ITS--ETC(U)
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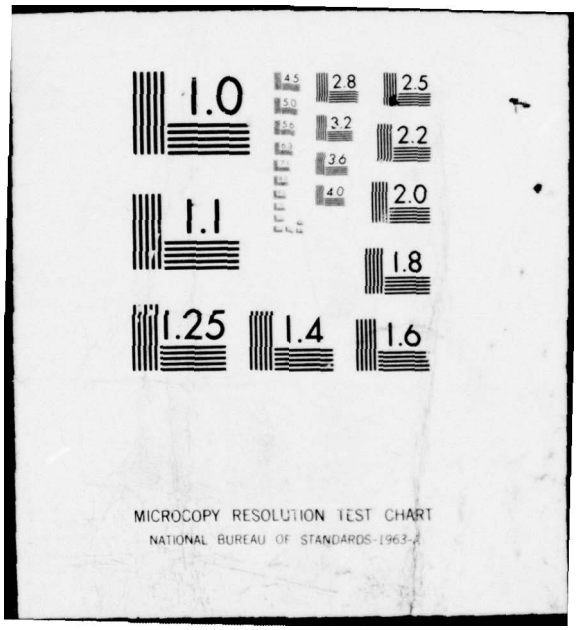
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1.25
1.4
1.6
1.8
2.0
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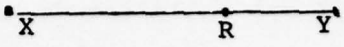
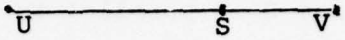
NATIONAL BUREAU OF STANDARDS-1963-A




XY = UV if XR = US and RY




(4) Corresponding parts of congruent triangles are equal.

LONG THEOREM EQLINE(L(X,Y), L(U,V)) FROM

CONG(Y,Z,X,R,W,U) ∧ DIFF3(X,Y,Z) ∧ DIFF3(U,V,W)

This theorem may construct new points Z and W. The DIFF3 predicates prove that these added points really form triangl

(5) The sides of an isosceles triangle are equal.

LISOC THEOREM EQLINE(L(X,Z), L(Y,Z)) FROM

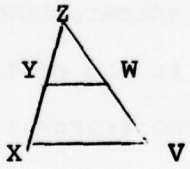
EQANG(A(Z,X,Y), A(X,Y,Z));

(6) A line parallel to the base of a triangle, which bisects one side, bisects the other.

LPARBASE THEOREM EQLINE(L(X,Y), L(Y,Z)) FROM

EQLINE(L(Z,W), L(W,V)) ∧ PARALLEL(L(Y,W), L(X,V)

∧ COLIN(X,Y,Z) ∧ COLIN(Z,W,V);




if YW ∥ XV and ZW ∥ WV ,

then XY = YZ

(7) Midpoints bisect lines.

LMID ASSERT EQLINE(L(X,MIDPT(X,Y)), L(Y,MIDPT(X,Y));

Additional ways in which lines may be proved equal are found in the section on coercions.

Angle Equality

(8) An angle is always equal to itself.

```
AREFLX ASSERT EQANG(X,X);
```

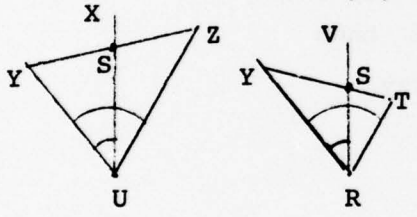
(9) Two angles are equal if they both equal a third.

```
ATRANS THEOREM EQANG(X,Y) FROM
      EQANG(X,Z) ∧ EQANG(Z,Y) ∧ DOWN30(X);
```

Again we will use DOWN30 to lower the use of transitivity.

(10) Differences of equal angles are equal.

```
APART THEOREM EQANG(A(X,Y,Z), A(V,R,I) FROM
      EQANG(A(X,U,Y), A(V,R,W)) ∧ EQANG(A(Y,U,Z), A(W,R,T))
      ∧ DOWN30(X) ∧ COLIN(U,J,X) ∧ COLIN(R,S,V)
      ∧ COLIN(Y,J,Z) ∧ COLIN(W,S,T)
```




if $\measuredangle YUZ = \measuredangle WRT$ and $\measuredangle YUX = \measuredangle VRW$

then $\measuredangle XUZ = \measuredangle VRT$

(11) Corresponding points of congruent triangles are equal.

```
ACONG THEOREM EQANG(A(X,Y,Z), A(U,R,W)) FROM
      CONG(X,Y,Z,U,V,W) ∧ DIFF3(X,Y,Z) ∧ DIFF3(U,V,W);
```

Once again DIFF3 is used to make sure that we have two triangles.

(12) The base angles of an isosceles triangle are equal.

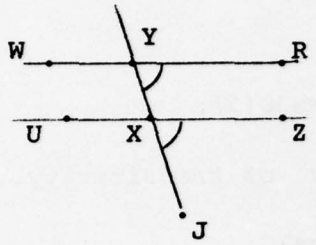
```
AISOSC THEOREM EQANG(A(X,Y,Z), A(Y,X,Z))
      FROM EQLINE(L(X,Z), L(Y,Z));
```

(13) Corresponding angles of parallel lines are equal.

ACOR THEOREM EQANG(A(R,Y,X), A(Z,X,J))

FROM PARALLEL(L(W,R), L(U,Z))

A COLIN(W,Y,R) ∧ COLIN(U,X,Z) ∧ COLIN(Y,X,J);



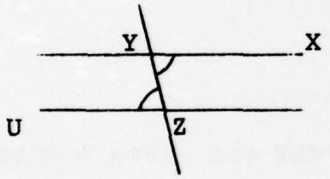

if WR ∥ UZ.

then ∠RYX = ∠ZXJ

(14) Alternate interior angles of parallel lines are equal.

ALTER1 THEOREM EQANG(A(X,Y,Z), A(U,Z,Y)) FROM

PARALLEL(L(Y,X), L(U,Z))



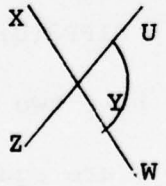

IF XY ∥ UZ then

∠XYZ = ∠UZY

(15) Vertical angles are equal.

VERTICAL THEOREM EQANG(A(X,Y,Z), A(U,Y,W))

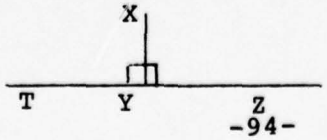
FROM COLIN(X,Y,W) ∧ COLIN(U,Y,Z)




(16) Two right angles add up to a straight line.

RTEQ THEOREM EQANG(A(X,Y,Z),RTANG) FROM EQANG(A(T,Y,X),RTANG)

∧ COLIN(T,Y,Z)



Congruence of Triangles

(17) A triangle is always congruent to itself.

```
CREF ASSERT CONG(X,Y,Z, X,Y,Z);
```

(18) Congruence by transitivity.

```
CRANS THEOREM CONG(X,Y,Z,U,V,W) FROM
 CONG(X,Y,Z,P,Q,R) ∧ CONG(P,Q,R,U,V,W)
 ∧ DOWN(X,Y,Z,U,V,W);
```

(19) The common ways to prove congruence, side-angle-angle:

```
CSAA THEOREM  CONG(X,Y,Z,U,V,W) FROM
 EQLINE(L(X,Y), L(U,V)) ∧ EQANG(A(X,Y,Z),A(U,V,W))
 ∧ EQANG(A(Y,Z,X), A(V,W,U);
```

(20) Side-side-side

```
CSSS THEOREM CONG(X,Y,Z,U,V,W) FROM
 EQLINE(L(X,Y), L(U,V)) ∧ EQLINE(L(Y,Z), L(V,W))
 ∧ EQLINE(L(X,Z), L(U,W));
```

(21) Side-angle-side

```
CSAS THEOREM CONG(X,Y,Z,U,V,W) FROM
 EQLINE(L(X,Y), L(U,V)) ∧ EQLINE(A(X,Y,Z),A(U,V,W))
 ∧ EQLINE(L(Y,Z),L(V,W));
```

Congruence is independent of the position of the viewer. Thus we may turn the page over or around. We need to include generators of the dihedral group to allow all possible ways of proving congruence.

```
(22)  CPERM THEOREM CONG(X,Y,Z,U,V,W)
            FROM CONG(Y,Z,X,V,W,U);

(23)  CFLIP THEOREM CONG(X,Y,Z,U,R,W)
            FROM CONG(X,Z,Y,U,W,V);
```

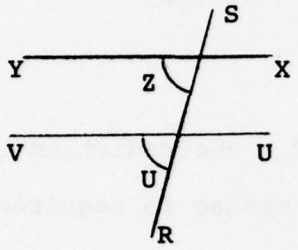
(24) The diagonal of a parallelogram, divides the parallelogram into two congruent triangles.

```
      CPARAL  THEOREM  CONG(X,Y,Z,X,U,Z) FROM
            PARALLGM(X,U,Z,Y);
```

## Parallel Line Theorems

(25) Two lines are parallel if corresponding angles are equal.

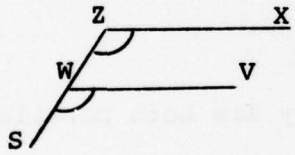
```
P1 THEOREM PARALLEL(L(X,Y), L(U,V)) FROM
     EQANG(A(Y,Z,W), A(V,W,R)) ∧ COLIN(X,Z,Y)
     ∧ COLIN(U,W,V) ∧ COLIN(S,Z,W) ∧ COLIN(Z,W,R);
```




XY ∥ VU

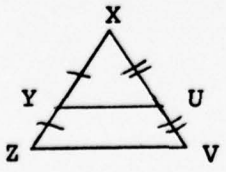
(26) We must also include the case in which the cross line doesn't pass the parallel lines.

```
P2 THEOREM PARALLEL(L(X,Z), L(V,W))
     FROM EQANG(A(Y,Z,W), A(V,W,S)) ∧ COLIN(Z W S);
```



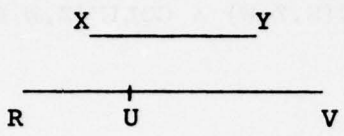

(28) A line which bisects both sides of a triangle is parallel to the base.

```
MIDPAR THEOREM PARALLEL(L(Y,U), L(Z,V)) FROM
     EQLINE(L(Z,Y),L(Y,X)) ∧ COLIN(Z,Y,X)
     ∧ EQLINE(L(V,U), L(U,X)) ∧ COLIN(V,U,X);
```



(28) Two lines are parallel if an extension of one is parallel to the second.

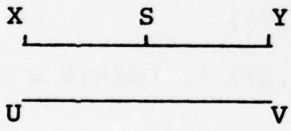
```
PEXT  THEOREM PARALLEL(L(X,Y , L(U,V))
      FROM PARALLEL(L(X,Y), L(R,V)) ∧ COLIN(R,U,V);
```




XY ∥ UV → XY ∥ RV

(29) Two line segments are parallel if a contradiction of one is parallel to the second. This clause is required since collinearity implies an order.

```
PBASE  THEOREM PARALLEL(L(X,Y), L(U,V))
       FROM PARALLEL(L(S,Y), L(U,V)) ∧ COLIN(X,S,Y);
```




XY ∥ UV ← SY ∥ UV .

(30) Two lines are parallel if they are both parallel to a third.

```
PTRANS THEOREM
 PARALLEL(X,Y) FROM PARALLEL(X,Z) ∧ PARALLEL(Z,Y);
```

Since lines are often proved parallel using transitivity, we don't need to include DOWN30.

These 30 clauses represent the geometric knowledge given to the outer theorem prover. In any particular run, only a subset of these clauses is used.

## 4.3 Coercicns

The inner theorem prover is given the following coercions:

Segment equality:

(1) LFLIP COER L(X,Y), L(Y,X);

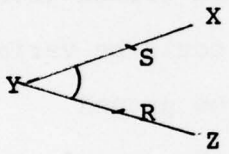
(2) LSYM COER EQLINE(X,Y), EQLINE(Y,X);

Angle equality:

(3) ASYM COER EQANG(X,Y), EQANG(Y,X);

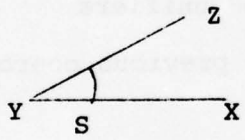
(4) Alternative names correspond to the same angle.

ANAME COER A(X,Y,Z), A(S,Y,R), COLIN(X,S,Y) ∧ COLIN(Z,R,Y);



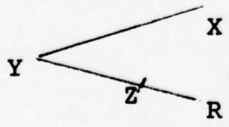

$\measuredangle XYZ = \measuredangle SYR$

(5) ANAME1 COER A(X,Y,Z), A(S,Y,Z), COLIN(X,S,Y);




XYZ = SYZ

(6) ANAME2 COER A(X,Y,Z), A(X,Y,R), COLIN(Y,Z,R);




(7) ANGR COER A(X,Y,Z), A(Z,Y,X);

Triangle congruence:

(8) CSYM COER CONG(X,Y,Z,U,V,W), CONG(U,V,W,X,Y,Z);

Each coercion was used in every run of the theorem prover. It is clear that the separation of clauses between inner and outer theorem prover was done in a somewhat arbitrary manner. The idea was that facts considered to be more "obvious" were given to the inner theorem prover.

## 4.4 Coercion Strategies

We use CPROCS to associate strategies with the coercions. These were chosen before any attempts were made to prove any theorems. The strategies were:

LFLIP: one of the two lines had to be ground (i.e. in the line L(x,y) neither x nor y could be variables

LSYM: one of the two lines had to be ground

ASYM: one of the two angles had to be ground

ANAME: no other solutions were found (i.e. this coercion could only be used if no unifiers had been found either using previous coercions or in the standard way)

ANAME1: no other solutions were found

ANAME2: no other solutions were found

ANGR: no other solutions were found

CSYM: no other solutions were found

Only one level of coercion could be used inside unification (MAXCOERS = 1).

These coercions are intended to "undo" the effects of canonical ordering. The ordering is itself done by partial evaluation. The ordering scheme is:

variables < points < logical functions

Variables are represented by numbers. Thus any two variables have an induced ordering. The BALM IFROMID function is used to order points.

L(X,Y) is ordered so that X < Y

A(X,Y,Z) is ordered so that X < Z

The use of canonical ordering is the major factor in the effectiveness of this program. A clause represents the entire set of preimages of the canonical form. Often this could be as many as 64 different alternatives (a clause mentioning 6 lines). Almost all resolvents thus become duplicates. The combination of canonical ordering and coercion does a huge amount of tree pruning without removing completeness.

It is interesting to note that we are using lexical properties rather than position in a model to order items. One of the uses of a model in Gelernter's scheme is thus unnecessary here.

Another "model" property, the evolution of "self-evident" truths is done by partial evaluation. We may reject any clause containing an instance of L(X,X) or A(X,Y,X), since two distinct points are required for a line and three for an angle. The predicate COLIN is partially evaluated. We have a list of all points which are collinear in the original problem. COLIN applied to three points simply checks if the points are on this list. Skolem functions such as MIDPT add new elements to this list. The DIFF3 predicate is also evaluated.

### 4.4.1 The Major Heuristics

Several predicates and functions are used to compute heuristic information.

L(X,Y) ↔ if X and Y are variables increase the complexity of a clause by 10

DOWN30 ↔ increase the complexity of a clause by 30

DOWN ↔ if any of the arguments of the predicate are variables increase the complexity of a clause by 30.

The complexity of a clause is computed in the following way.

### 4.4.2 Complexity Function

(1) complexity starts at zero

(2) If the clause was produced by resolving with a special hypothesis of the problem, lower the complexity by 50

(3) For each literal which is unifiable with a special hypothesis, lower the complexity by 20;
for each other literal, increase the complexity by 10

(4) Add twice the number of variables in each literal

(5) Add 20 for each CONG literal

(6) Add 5 for each nonground literal

(7) For literals which have PE's associated to their predicate don't use 3-6; instead add 5 + 4 * number of variables

(8) Reject clauses with more than 3 PARALLEL predicates

(9) Reject clauses longer than 10 literals.

4.4.3 Actual Input

Actual input to the program consists of

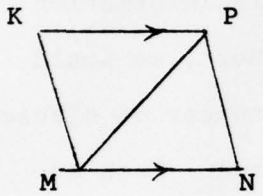
(1) the special hypotheses of the problem

(2) a list of collinear points as the value of the variable COLINER

(3) a list of the names of the special hypotheses

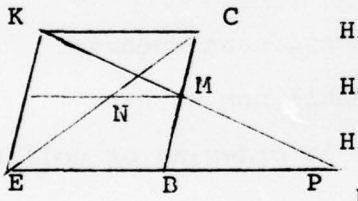
(4) the initial goal to be proved.

For example the G2 problem input is:




```
H1: ASSERT EQLINE(L(P,K),L(N,M));
H2: ASSERT PARALLEL(L(P,K),L(M,N));
COLINER = NIL;
HYP = (H1,H2);
PROVE(EQLINE(L(P,M),L(K,N));
```

A somewhat more elaborate problem is G5:




```
H1: ASSERT PARALLEL(L(K,C),L(E,D));
H2: ASSERT EQLINE(L(E,N),L(N,C));
H3: EQLINE(L(K,M), L(M,D));
HYP = ' (H1 H2 H3);
```

```
COLINER = ' ( ((E)(P)(K)) ((K)(P)(E)) ((P)(N)(M)) ((M)(N)(P))
              ((K)(M)(D)) ((D)(M)(K)) ((E)(B)(D)) ((D)(B)(E))
              ((C)(M)(B)) ((B)(M)(C)) ((E)(N)(C)) ((C)(N)(E)) );
PROVE(EQLINE(L(E,P), L(P,K));
```

## 4.5 Point Constructions

The geometry program which we have described in the previous section, lacks a very important geometric fact; that any pair of distinct nonparallel lines intersect in exactly one point. Given this fact, a program could introduce a new point defined as the intersection of a pair of lines. In this section, we will show how this fact may be added to the original system.

One possible way to incorporate this information would be to use an axiomatic approach. Here, we would define a point predicate and then add a number of clauses specifying how points and lines are related. Such an approach leads to very many new clauses. These represent information of a very different nature from that in the Sections 4.1-4.3. For example to say that a point is either to the left or right of a line is far more basic than to say that two triangles are congruent whenever corresponding sides are equal. Until now we have represented simple point ideas such as ordering or collinearity entirely by partial evaluations; we chose to encode this new fact in the same way.

We first introduce two new constants, R and I. We will write <R,P,Q> when we wish to represent an undetermined point on the line joining P and Q, with the order of the points being P,Q,X.

Similarly <I,P,Q> will represent a point inside the segment PQ:


<I,P,Q> is on this segment

Next we introduce a six place logical function

INTER(TYPE1,TYPE2,P11,P12,P21,P22)

This will represent the point which is both of the triples

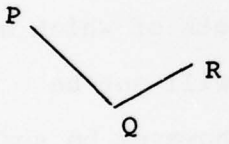
<TYPE1,P11,P12>

and

<TYPE2,P21,P22> .

If the lines P11 P12 and P21 P22 are parallel then INTER will be undefined.

The use of INTER raises a new issue: that of multiple names for the same point, e.g. in the picture:




INTER(R,R,P,Q,R,Q) is also called Q.

We will therefore associate a partial evaluator with INTER, which will reduce INTER of 6 constants to a point name when this is possible. Of course we will need a coercion to unevaluate INTER. This coercion will allow

INTER(R,X,P,Q,Y,Z) to unify with a point B

with the added conditions:

$$X = R \wedge COLIN(Y,Z,B)$$

or

$$X = I \wedge COLIN(Y,B,Z)$$

The COLIN predicate needs to be changed as well. COLIN(P,Q,X) evaluates to TRUE with the condition

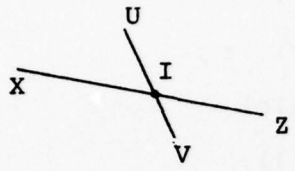
$$X = INTER(R,U,P,Q,R,W)$$

COLIN(P,Q,INTER(R,U,B,E,V,W)) evaluates to true with the conditions $U = R$, $V = P$, $W = Q$, $PQ$ not parallel to $BE$.

The most pronounced change added to our system is the need to detect when lines are not parallel. To resolve this we will introduce additional information from the diagram. We wish to point out that this is still nonnumeric information. We will give the program a list, each of whose elements will be a list of line segments. The interpretation will be that any two segments which are on the same list "look parallel". The program will not attempt to construct a point as the intersection of two lines both of which are on the same list. Being on the same list will not be sufficient to prove parallelism. It will however be sufficient to prevent constructions. We introduce a new PE predicate CROSSES(X,Y,U,V) ↔ XY does not look parallel to UV.

Finally we want to add two new clauses. Since the coercions are present, these clauses are not necessary for completeness, yet they increase the efficiency. Both clauses will state that a line is parallel to a subsegment bounded by a second line.

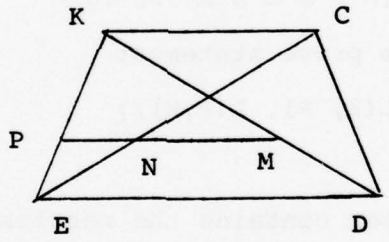
```
PCR1 THEOREM PARALLEL(L(X,INTER(TP1,TP2,U,V,X,Z)),
                      L(X,Z)) FROM CROSSES(X,Z,U,V);

PCR2 THEOREM PARALLEL(L(X,INTER(TP1,TP2,X,Z,Y,V)),
                      L(X,Z)) FROM CROSSES(X,Z,U,V);
```




These clauses state that XI is parallel to XZ, where I is the intersection of UV and XZ.

A sample input to the extended geometry program is the following:

Given the problem




Special hypotheses KC ∥ ED, EN = NC, KM = MD

Prove EP = PK

Computer Input:

```
H1  ASSERT PARALLEL(L(K,C),L(E,D));
H2  ASSERT EQLINE(L(E,N),L(N,C));
H3  ASSERT EQLINE(L(K,M),L(M,D));
```

[continued]

```
* list of names of special hypotheses
HYP = = (H1 H2 H3);
* list of   collinear points
COLINER = (((E)(P)(K)) ((K)(P)(E))
           ((P)(N)(M)) ((M)(N)(P))
           ((K)(M)(D)) ((D)(M)(K))
           ((E)(N)(C)) ((C)(N)(E)) );
* list of lines which look parallel
LOOKSP = ' (((K C)(N M)(P M)(E D)));
* list of alternative names
PNAMES = ' (((R R K P D E) E)
            ((R R C N D E) E)
            ((I R E K M N) P)
            ((I I E C P M) N)
            ((R R E D K M) D) );
* finally the prove statement
PROVE(EQLINE(L(E, P), L(P,K));
```

The next chapter contains the results of running both the original and the extended geometry program on a number of simple problems.

# CHAPTER 5

## EXPERIMENTAL RESULTS

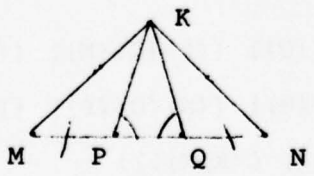
In order to test the effectiveness of the programs described in Chapter 4, we performed a number of experiments. We chose six different geometric problems, and did a number of variations of each. In this chapter, each of these experiments is analyzed in detail. For each problem we will give a picture, and a list of axioms. We will show the refutation tree and indicate where the coercion scheme eliminated resolutions.

The six geometric problems which we named G0,G1,G2,G3,G4,G5 have all been proved previously by machine.

G0, the simplest of the problems, was introduced by Ira Goldstein [13]. Problems G1 through G5 are due to Gelernter [10,11]. Examples G0 through G4 were run with the basic system. Variations of G5 were run with both the basic and extended system.

### 5.1 The G0 Problem (Problem 32, June 1969, N.Y. State Regents)

Given segment MPQN with MP = QN and angle KPQ = angle KQP prove that segment MK = segment KN.



To specify the input to this problem, we must first indicate which clauses and coercions were used.

Coercions:

| | |
|---|---|
| ANGR, ANAME1, ANAM2 | alternative names of angles are equivalent |
| CSYM, LSYM | CONG, EQLINE, EQANG are symmetric relations |
| ASYM, FLIP | the line XY is equivalent to the line YX |

Clauses:

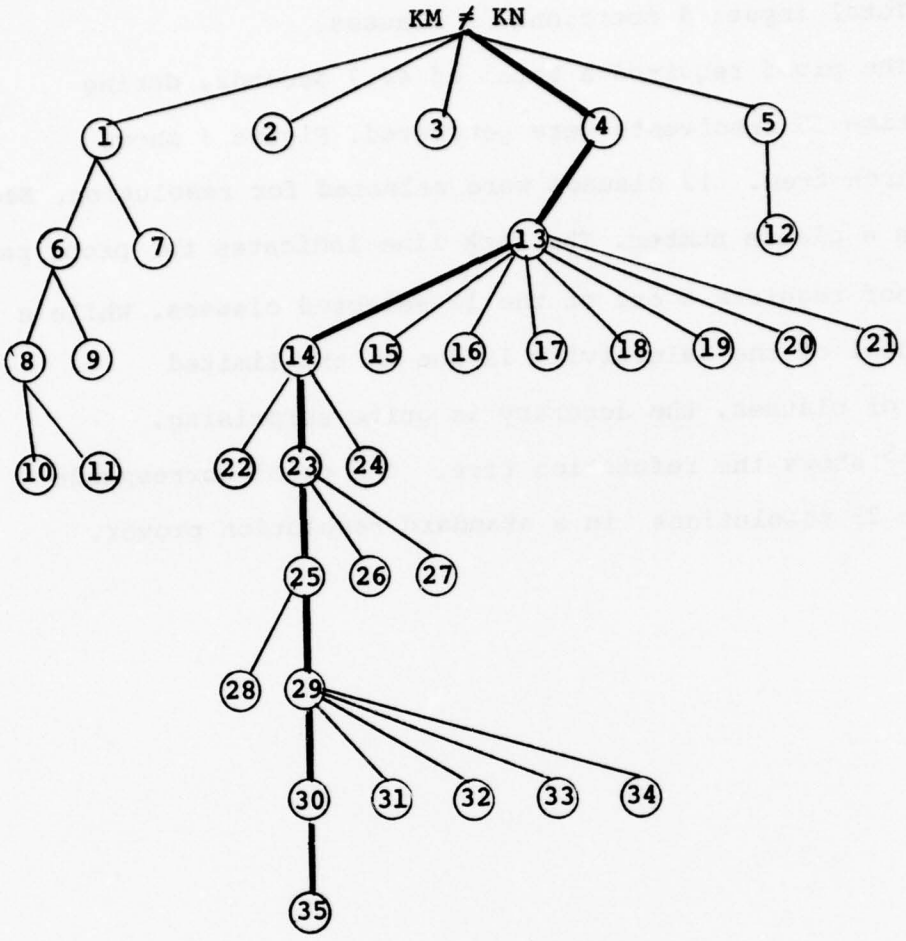
| | |
|---|---|
| AST | two angles are equal if both are straight |
| APART | parts of equal angles are equal |
| ATRANS, AREFLX | angle equality is transitive and reflexive |
| LISOC | sides of isosceles triangles are equal |
| LCONG | corresponding sides of congruent triangles are equal |
| CSAS | congruence by side, angle, side. |

Input:

```
H1  ASSERT EQANG(A(K,P,Q), A(P,Q,K));
H2  ASSERT EQLINE(L(P,M), L(N,Q));
HYP = '(H1 H2);
COLINER = '(((M)(P)(Q)) ((P)(Q)(N)) ((M)(P)(Q)(N))
           ((Q)(P)(M)) ((N)(Q)(P)) ((N)(Q)(P)(M)) );
PROVE (EQLINE(L(K,M), L(K,N)));
```

Total input: 8 coercions, 9 clauses.

The proof required a total of 44.7 seconds, during which time 35 resolvents were generated. Figure 4 shows the search tree. 12 clauses were selected for resolution. Each node is a clause number. The dark line indicates the proof path. The proof requires 8 out of the 12 selected clauses. While a great deal of the selectivity is due to the limited number of clauses, the accuracy is quite surprising. Figure 5 shows the refutation tree. The proof corresponds to some 23 resolutions in a standard resolution prover.




44.7 seconds

12 clauses selected

35 resolvents

Figure 4. G0 (Variation 1). Search Tree.

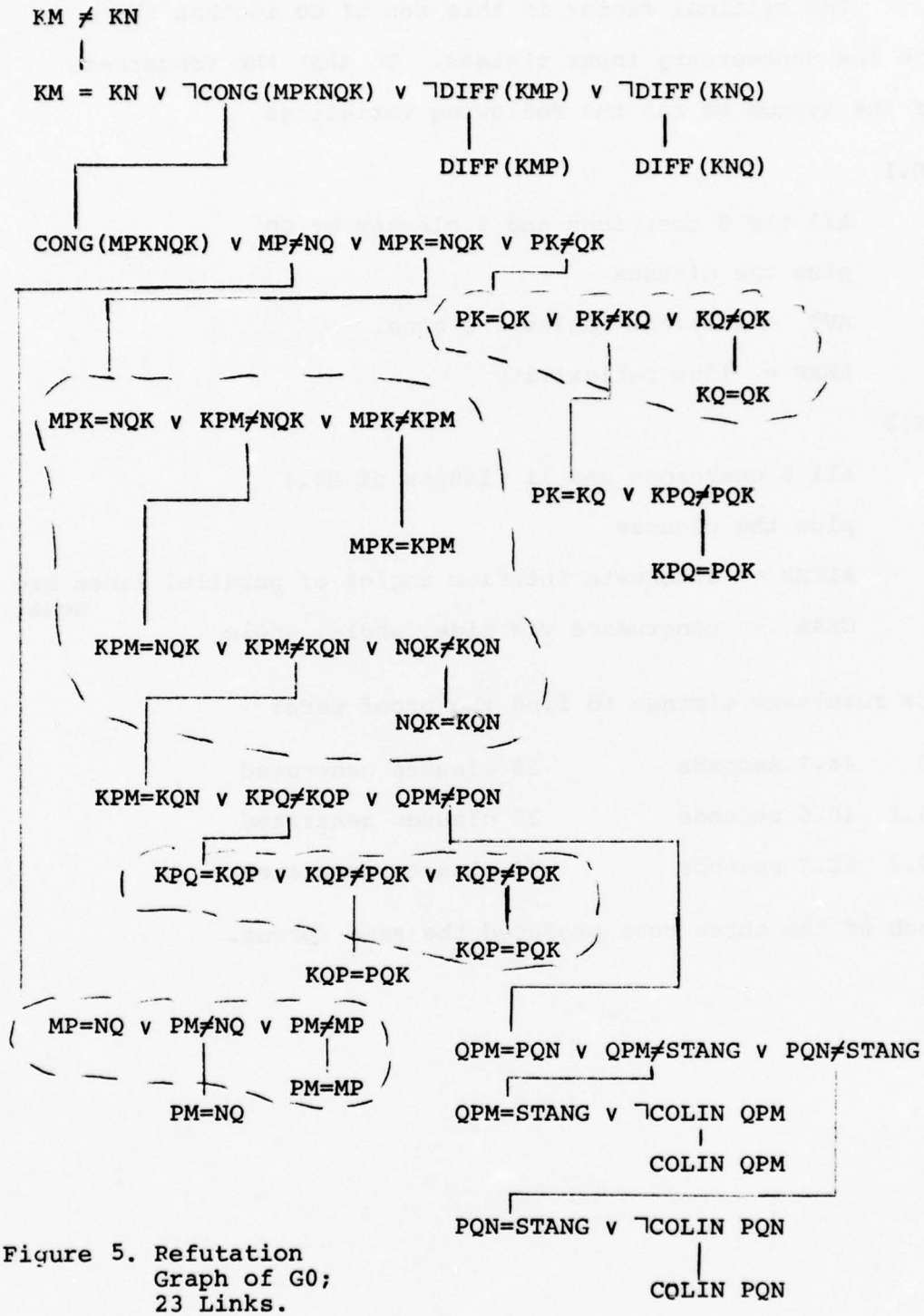



Figure 5. Refutation Graph of G0; 23 Links.

The critical factor in this run of G0 is that there are few unnecessary input clauses. To test the robustness of the system we ran the following variations.

G0.1

All the 8 coercions and 9 clauses of G0

plus the clauses

AVT ↔ vertical angles are equal

LREF ↔ line reflexivity

G0.2

All 8 coercions and 11 clauses of G0.1

plus the clauses

ALTER ↔ alternate interior angles of parallel lines are equal

CSAA ↔ congruence via side, angle, angle

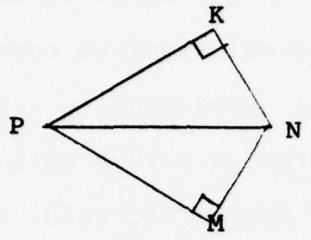
The resultant timings to find the proof were:

| | | |
|---|---|---|
| G0 | 44.7 seconds | 35 clauses generated |
| G0.1 | 45.6 seconds | 35 clauses generated |
| G0.2 | 52.3 seconds | 40 clauses generated |

Each of the three runs produced the same proof.

## 5.2 The G1 Problem (Gelernter, Problem 1)

Given that angle KPN = angle NPM, and that angles PKN and PMN are both right angles, prove that segment KN = segment NM.




Input to the theorem prover consisted of the 8 coercions used for G0 plus the following clauses:

| | |
|---|---|
| AST | straight angles are equal |
| APART | parts of equal angles are equal |
| ATRANS, AREFLEX | angle equality is transtivie and reflexive |
| LREF | line equality is reflexive |
| LISOC | sides of an isosceles triangle are equal |
| LCONG | corresponding sides of congruent triangles are equal |
| CSAA | congruence by side, angle, angle |
| CSAS | congruence by side, angle, side. |

Only LCONG, CSAA, LREF are used in the proof.
The special input for G1 was:

```
H1  ASSERT  EQANG(A(K,P,N), A(N,P,M));
H2  ASSERT  EQANG(A(P,K,N), RTANG);
H3  ASSERT  EQANG(A(P,M,N), RTANG);
```

[continued]

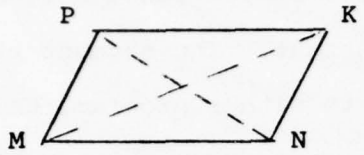
```
HYP =  '(H1 H2 H3);
COLINER = NIL;
PROVE(EQLINE(L(K,N), L(N,M));
```

Gelernter's system took approximately 25 seconds to produce a 10 step proof which is equivalent to the resolution proof. The seven step coercion proof required some 86.5 seconds selecting 12 clauses and producing 58 resolvents. Gelernter's program was run on a 7090 while the Dilemma system runs on the considerably faster CDC 6600. At the top level the proof consisted of:

```
     EQLINE(L(K,N), L(N,M))
                    \
                     LCONG
(5)  ¬CONG(NXK NYM) v ¬DIFF(K N X) v ¬DIFF(M N Y)
                \
                 CSAA
(8)  ¬EQLINE(L(NX), L(NY)) v ¬EQANG(A(KXN), A(NYM))
           v ¬EQANG(A(XKN), A(YMN)) v ¬DIFF(KNX)
           v ¬DIFF(MNY)
                      \
                       LREF
(35) ¬EQANG(A(KXN), A(NXM)) v ¬EQANG(A(XKN), A(XMN))
           v ¬DIFF(KNX) v ¬DIFF(MNX)
                        \
                         H1
(46) ¬EQANG(A(PKN), A(PMN))
                        \
                         ATRANS
(52) ¬EQANG(A(PKN),X) v ¬EQANG(X,A(PMN))
                        \
                         H2
(53) ¬EQANG(RTANG,A(PMN))
                        \
                         H3
(58)  □
```

5.3 The G2 Problem (Gelernter, Problem 2)

Given the quadrilateral PKNM with segment PK equal to segment NM and PK parallel to NM, prove segment PM = segment KN.




Inputs: the 8 coercions used in G0 and G1 plus

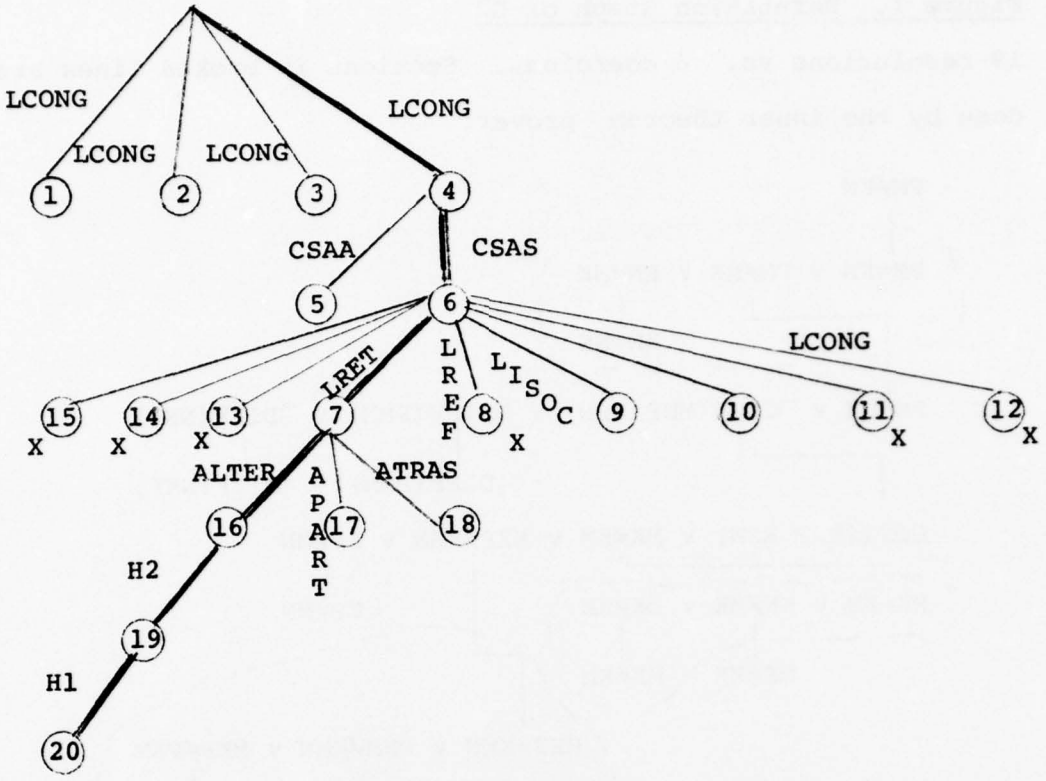
| | |
|---|---|
| ALTER1 | alternate interior angles of parallel lines are equal |
| AST | all straight angles are angle |
| APART | parts of equal angles are equal |
| ATRANS, AREF | angle equality is transitive and reflexive |
| LREF | line equality is reflexive |
| LISOC | sides of an isosceles triangle are equal |
| LCONG | corresponding sides of congruent triangles are equal |
| CSAA | congruence by side, angle, angle |
| CSAS | congruence by side, angle, side |

```
H1  ASSERT EQLINE(L(P,K), L(M,N));
H2  ASSERT PARALL (L(P,K), L(M,N));
PROVE(EQLINE(L(P,M), L(K,N));
```

The search for a proof selected 6 clauses, which produced 20 resolvents. The proof is 19 levels deep for resolution and 6 levels for coercion. The

coercion system found the proof in 10.2 seconds. This problem is by far the easiest of all the geometry examples. It is interesting to see the difference in performance between our system and Gelernter's on this problem. Since Gelernter's system did not have Skolem functions, each line segment had to be a named constant. The segment PN (the diagonal line) was not given to either program. Gelernter's program after about a minute gave up on this problem, went into its construction phase, added PN, and then after another 4 minutes produced the proof. Our program used the MK diagonal. Constructions such as PN are done automatically in our system, as the result of Skolem functions.

Another observation is that at one point our problem resolves $MK \neq KM$ with $X = X$ via the coercion $MK = KM$. This is redundant, the subtree shows that we could just as well resolve the literal away with a unary resolvent. Figure 6 gives the search tree; Figure 7 the refutation tree for G2.

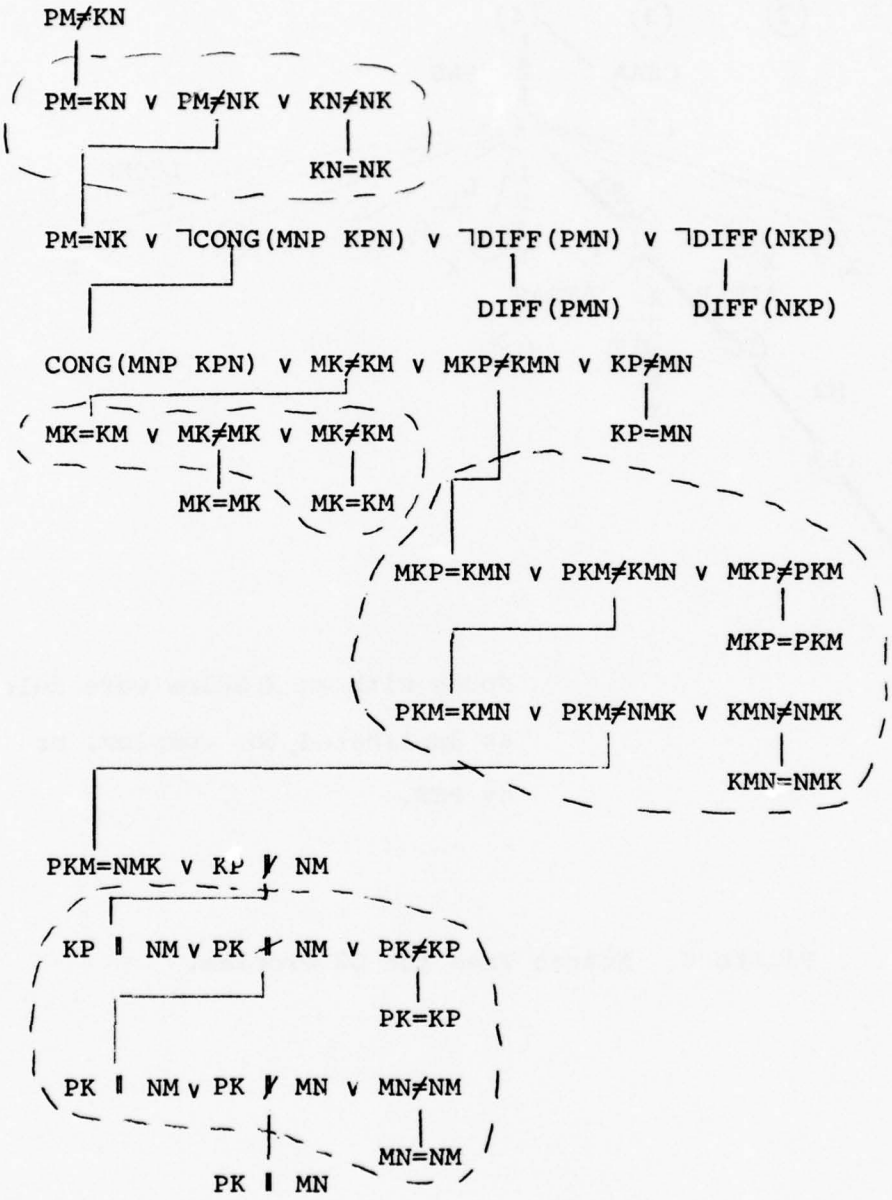



Nodes with an X below were deleted as duplicated, too complex, or by PES.

Figure 6. Search Tree for G2 Problem.

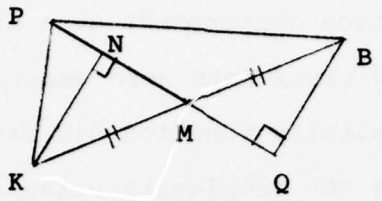
Figure 7. Refutation Graph of G2

19 resolutions vs. ȯ coercions. Sections in broken lines are done by the inner theorem prover.



## 5.4 The G3 Problem

Given the following figure, with KM = MB, and angles PNK and MQB both equal to 90 degrees, prove that KN = BQ.




Inputs: the 8 basic coercions and 10 clauses of G2 as well as:

| | |
|---|---|
| VERTICAL | vertical angles are equal |
| RTEQ | two right angles form a straight angle |

```
H1 ASSERT EQLINE(L(K,M), L(M,B));
H2 ASSERT EQANG(A(P,N,K), RTANG);
H3 ASSERT EQANG(A(M,Q,B), RTANG);
COLINER = '(((P)(N)(M)) ((N)(M)(Q)) ((K)(M)(B))
           ((M)(N)(P)) ((Q)(M)(N)) ((B)(M)(K)) );
PROVE(L(K,N), L(B,Q))
```

Gelernter ran this problem twice; first on his original system which produced a somewhat redundant proof after 8 minutes and then on an extended system which produced a proof after 1 minute. The extensions consisted of two heuristics. The first allowed goals to be expanded out of order. Identities, equalities between angles, and various other goals

could usually be satisfied in one step. Gelernter's heuristic gave preference to such goals. The second heuristic was a complexity measure.

Our system solved this problem in 4.5 minutes. It found an 8 level coercion which corresponds to a 20 level resolution proof. A total of 144 resolvents were generated from 34 selected clauses. The complexity function did rather poorly here. For comparison we give the complexity value for each resolvent in the proofs for G1, G2, and G3.

| G1 | | G2 | | G3 | |
|---|---|---|---|---|---|
| Clause | Value | Clause | Value | Clause | Value |
| 5 | +75 | 4 | +75 | 4 | +75 |
| 8 | +66 | 6 | +66 | 24 | +66 |
| 35 | +46 | 7 | +14 | 51 | 6 |
| 46 | -17 | 16 | -38 | 65 | +33 |
| 52 | +47 | 19 | -69 | 67 | +77 |
| 53 | -69 | 20 | - ∞ | 140 | -13 |
| 58 | - ∞ | | | 141 | -58 |
| | | | | 144 | - ∞ |

This table shows that the complexity measure did badly in G3 on clauses 65 and 67 which accounts for the large amount of time necessary to find the proof. Clause 67 consists of:

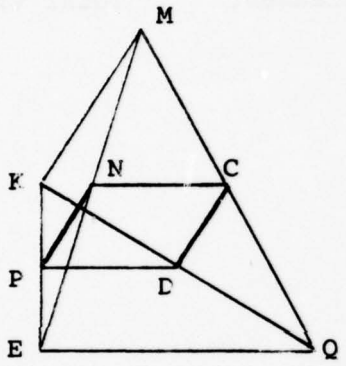
$$\neg(\text{angle}(K\ N\ M) = X \vee X = \text{angle}(M\ Q\ B))$$

This clause was produced by an application of angle transitivity which is given a penalty each time it is used.

## 5.5 The G4 Problem

Gelernter introduced additional extensions to his program in 1960. The first of these extensions allowed that any two names for an angle could be recognized as equal solely by using the diagram. The second extension was the point generator which has been described above. Using these extensions, Gelernter's system solved the G4 and G5 problems, requiring 1 minute on G4 and 31 minutes on G5.

G4




Given: EP = PK
EN = NM
MC = CQ
KD = DQ

Prove: NC ∥ PD

Inputs: the 8 coercions and 12 clauses of G3 plus

```
PSYM COER PARALLEL(X,Y), PARALLEL(Y,X);
```

PARTRANS — parallelism is transitive

MIDPAR — the line which bisects the sides of a triangle is parallel to the base.

```
H1 ASSERT EQLINE(L(E,P), L(P,K));
H2 ASSERT EQLINE(L(E,N), L(N,M));
H3 ASSERT EQLINE(L(C,Q), L(M,C));
H4 ASSERT EQLINE(L(K,D), L(D,Q));
HYP = '(H1 H2 H3 H4);
```

```
COLINER = '( ((E)(P)(K)) ((K)(P)(E))
             ((E)(N)(M)) ((M)(N)(E))
             ((M)(C)(Q)) ((Q)(C)(M))
             ((K)(D)(Q)) ((Q)(D)(K)) );
PROVE(PARALLEL(L(N,C), L(P,D));
```

The proof found was 7 coercions long, and corresponded to a 20 level resolution proof. A total of 331 resolvents were produced from 32 selected clauses. Total time was 25.7 seconds.

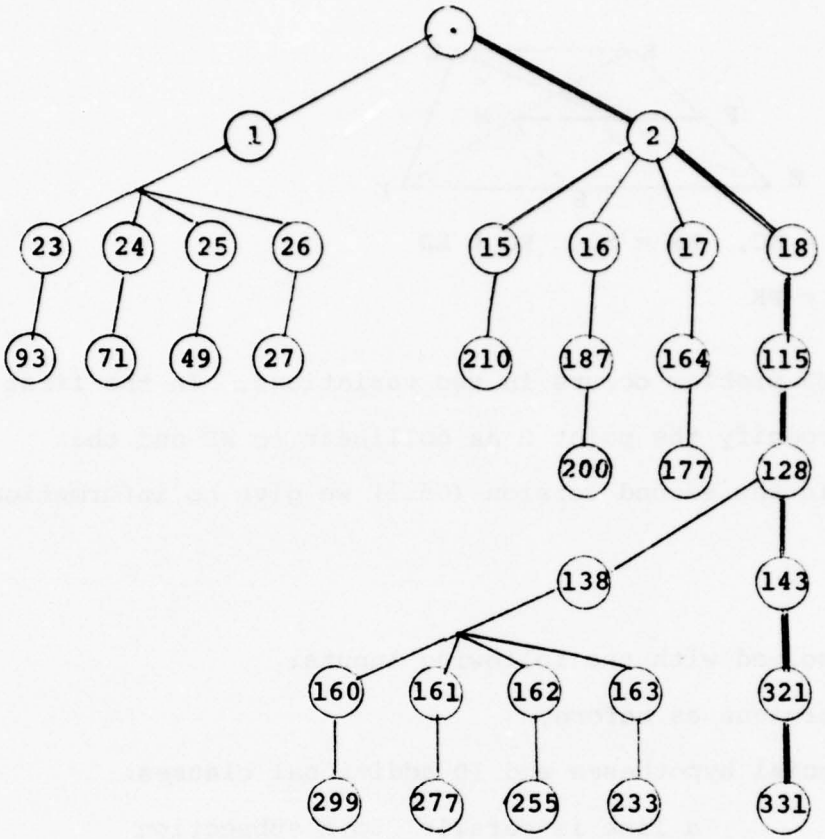
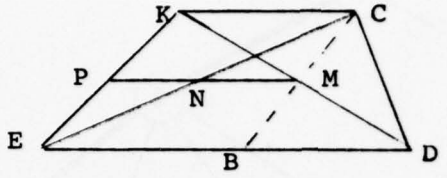



Figure 8. G4 Search Tree. Only nodes which have complexity 42 or less are shown.

## 5.6 The G5 Problem




Given: EN = NC, KM = MD, KC ∥ ED

Prove: EP = PK

The G5 problem occurs in two variations. In the first (G5.0) we specify the point B as collinear to ED and that BM = MC. In the second version (G5.1) we give no information about B.

G5.0 was solved with the following inputs:

8 coercions as before,

4 special hypotheses and 10 additional clauses:

| | |
|---|---|
| PEND | a line is parallel to a subsection |
| PBASE | a line is parallel to another if the other is parallel to a subsection |
| PEXT | a line is parallel to its extensions |
| PARTRANS | parallelism is transitive |

MIDPAR, ALTER1, VERTICAL, LCONG, LPARBASE, CSAA.

The proof required 8 coercions or 11 resolutions and was found in 153 seconds. The search reached 100 clauses in size.

The G5.1 problem has only been solved by Gelernter's extended system. The necessary construction prevented all the other geometry programs which we have described from solving it.

The G5.1 problem required 677 seconds using the extended system. 571 resolvents were produced. Figure 8 gives the final proof tree. The proof used 16 coercions, or 27 resolutions, and 88 clauses were selected.

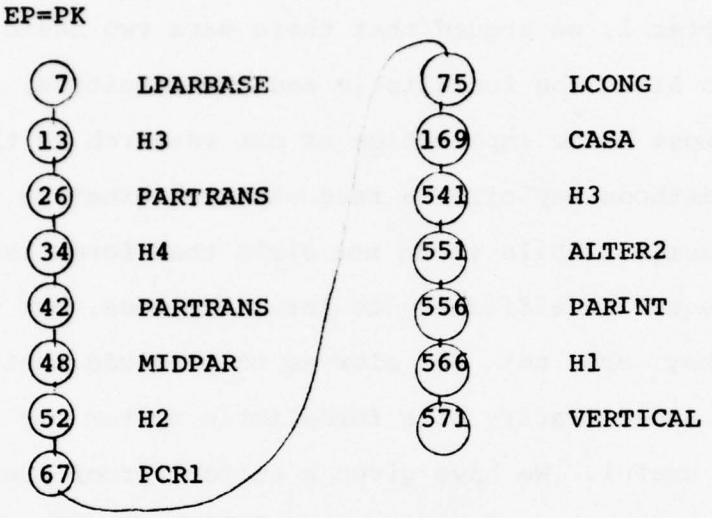



Figure 8. Proof of G5.1

Our results are summarized in Table III.

| Problem | Number of Input Clauses | | Proof | | Final Pool Size |
|---|---|---|---|---|---|
| | Coercions | Clauses | Coercions | Resolutions | |
| G0 | 8 | 9 | 8 | 23 | 13 |
| G0.1 | 8 | 11 | 8 | 23 | 14 |
| G0.2 | 8 | 13 | 8 | 23 | 16 |
| G1 | 8 | 12 | 7 | 14 | 58 |
| G | 8 | 12 | 6 | 19 | 7 |
| G3 | 8 | 15 | 8 | 20 | 26 |
| G4 | 8 | 19 | 7 | 20 | 103 |
| G5.0 | 8 | 14 | 8 | 11 | 100 |
| G5.1 * | 8 | 13 | 16 | 27 | 13 |

* Using extended system.

# CHAPTER 6

# CONCLUSIONS

## 6.1 Formalistic Methods

In Chapter 1, we argued that there were two basic approaches to AI -- the formalistic and the intuitive. Perhaps the most basic implication of our research is that formalistic methods may offer a reasonable alternative to intuitive schemes. While we do not claim that formalistic methods are equal in efficiency to intuitive ones, our results show that they are not so slow as to preclude their use. In fact, the added clarity of a formalistic system may make it more useful. We have given a uniform proof procedure and shown that it may be applied to group theory, logical puzzles, and geometry. We believe that any intuitive program which could solve problems in each of these diverse areas would be very awkward.

## 6.2 Hierarchical Theorem Proving

The figures in Chapter 2 show that on a limited set of theorems, hierarchical theorem proving is more effective than resolution. The basic notion of replacing the requirement of identity with proofs of equivalence inside unification is widely applicable. While we have only used it in Horn clause resolution,

Theorem 5 shows that it could be used in any Herbrand type proof procedure. For example hierarchical notions could be applied in linked conjuncts, model elimination, and all the other unification based schemes.

The coercion mechanism may not be the best way to implement Theorem 5. We expect that difference operations as used in GPS could be significantly more effective. Or one could allow an inner theorem prover, which contained an algebraic simplifier. Certainly the notion of an inner theorem prover could be extended to include additional, even more internal, theorem provers.

## 6.3 Partial Evaluation and Heuristics

The geometry program shows that partial evaluation is a useful adjunct in a theorem prover. The use of partial evaluation changed a large number of distinct clauses into the same canonical form. It is doubtful if the geometry proofs could have been achieved without such pruning. At the same time, we saw that it was necessary to unevaluate expressions. The hierarchical method allowed just the sort of unevaluation which was needed. We believe that the use of coercion suggests a reasonable way out of the difficulties encountered in QA3 and other partial evaluation systems. Also, the use of predicates to set heuristic switches was

very useful. Such a way of adding heuristics to a uniform proof procedure is natural. The claim that uniform procedures cannot make use of domain dependent information is false.

## 6.4 The Geometry Program

Finally we come to the most immediate conclusion of this research: that it is possible to use a resolution type theorem prover to solve problems in elementary geometry. When we began this work, such a result would have been considered unlikely. Many AI workers seem to believe that formal theorem provers are unsuitable for such problems.

## 6.5 Future Possibilities

The completeness theorem given in Chapter 2 states that two level theorem proving may be used with any set of clauses which has a refutation tree. The Dilemma language allows only clauses which are in Horn form. Horn sets are a proper subset of the clauses with a refutation tree, as shown by the following example.

$$S = \left\{ \begin{array}{l} P(a,x) \vee P(b,x) \vee \neg P(c,x) \vee \neg P(d,x) \\ \neg P(a,x) \\ \neg P(b,x) \\ P(c,x) \\ P(d,x) \end{array} \right\}$$

This set is unsatisfiable, has a refutation tree, and cannot be renamed into Horn clause form. We will now describe a splitting algorithm (Chang [ 5 ]), which will change an arbitrary set of clauses into several sets of Horn clauses. We will then use S to give an example of the use of this technique.

Chang's Algorithm:

Let $R = R_1 \cup A \vee B$ be a set of clauses.

(1) Prove that $R_1 \cup A$ is unsatisfiable.

If there were n distinct occurrences of instances of A in the refutation let $\sigma_1 \ldots \sigma_n$ be mgus such that the ith instance of A is $A\sigma_i$.

(2) R is unsatisfiable if

$R_1 \cup B\sigma_1 \cup B\sigma_2 \ldots \cup B\sigma_n$ is unsatisfiable.

In the example, let $A = P(a,x)$

$B = P(b,x) \vee \neg P(c,x) \vee \neg P(d,x)$

$R_1 = S - A \vee B$

Step 1 consists of

$P(a,x)$

$-P(a,x) \quad P(c,x)$

$-P(b,x) \quad P(d,x)$

which allows only one resolution, uses only one instance of $P(a,x)$, and involves the null substitution $\sigma$.

Step 2 requires proving that $R_1 \cup B\sigma$ is unsatisfiable.

This is a standard Horn clause problem and is easily solved.

This algorithm could be implemented by writing a program which does the splitting and then calls our system in order to solve each subproblem. A new level of hierarchical structure could be provided by modifying our system to produce the refutations of subproblems simultaneously.

Dilemma itself could be extended in several ways. First, we could add a sense of time. Mathematical systems are static; once proved, a theorem stays true. In many situations, however, the truth of a statement changes over time. For example, to a robot a set of theorems could indicate where things are in the current environment. As objects are moved, these theorems would change. We believe that a dynamic component could be added to Dilemma in a way that would make it possible to treat robot control problems conveniently. Another extension would be to add other special inner theorem provers. While the coercion mechanism works well for the equality relation, other relations such as partial ordering or set equality could be added. An open question is whether U-g -resolution can be extended in efficient ways to build in such relations. We expect that additional information about problem solving would be found by writing other programs in Dilemma. For example, a program which does symbolic integration. Here a whole new class of heuristics would be necessary. Possibly an inner theorem prover which knows about algebraic simplification would be required. Finally we could build special hardware.

The Dilemma language is highly parallel. Perhaps a large number of micro processors, each computing a generalized resolution, would lead to a very fast Dilemma machine.

Considering the theorems of Chapter 2, we see that there are a number of unresolved theoretical issues. What happens to completeness when the refutation graph contains cycles? Since these cycles arise from merges, can we restrict how merges occur, for example, to merges on only certain predicate symbols? We have shown that the sets of clauses which are characterized by input proofs, unit proofs, or refutation trees are the same. These are the easy problems for a number of algorithms. Can we generalize this to a complexity measure, i.e. an unsatisfiable set of clauses is of complexity M if the minimum number of merges (i.e. cycles in a refutation graph) is M? (see Kowalski [23]).

Finally, as we pointed out in Chapter 1, our work is one more step in the direction started by McCarthy toward building programs which would learn. We hope that the Dilemma language and our ideas of hierarchical theorem proving may contribute to the design of an underlying representation of information by a machine. The larger, more global, question raised by our work is how a machine which is given some instruction can transform that instruction into optimal Dilemma procedures.

## APPENDIX I

## IMPLEMENTATION DETAILS AND LISTING OF THE COMPILER

The implementation is divided into four sections: basic utilities, input parser, unification, post unification.

### Basic Utilities

| | |
|---|---|
| APPEND(L1,L2) | append LIST L2 to LIST L1<br>L1 is changed. Result is in the new L1. |
| RENAME(L1,I) | L1 is a list of literals. I is a number.<br>All variables of L1 are renamed. The left-<br>most variable gets the name I+1. |
| NAMELIT(lit) | does the renaming. A loop in RENAME calls<br>this procedure. |
| LOOKUP(item,list) | list is a list of pairs ((x,y)(z,v)).<br>If item is equal to the head of a sublist,<br>the value of lookup is the tail. Bindings are<br>just such lists. |
| SUB1(C,B) | Apply binding list B to clause C.<br>Change the clause in place. |
| MAKNAME(cℓ) | Change the variables in a clause to Z1,Z2,...Z12. |
| DELETE(item,list) | Delete 11 copies of item from list.<br>Change list. |
| MAPR(list,proc) | Like map x but constructs the new list (in place) |
| PRINTHIST(cℓ) | cℓ clause vector. Print history of the<br>deduction of cℓ. |
| COPX | A high speed version of the BALM copy. |
| SUBINPLA(lit,B) | Apply binding list B to literal lit. |
| SUBER(lit,item,replace) | Change item to replace in literal. |

Partial evaluation works inside out (it is the LISP EVAL function). Some tricks are done with the variables DEPTH and MERIT. They look like global variables (but are actually copies).

CLEARSYS drops the BALM system (it gets about 7K additional space)

RTM is the run time monitor; the details of built-in equality are here.

Input Parser

Mostly input is a set of macro transformations. REMCO (remove comma) changes BALM tree structures into Dilemma objects.

Unification

A unifier is a pair (B,L); B = binding list, L = additional literals (added via coercions)

PUSHUNIT is used to combine two unifiers

MGU1 applies a coercion

COPYPAIR returns the pair of equal or equivalent items associated with a coercion (it only does a copy if the coercion passes the possible test).

Post Unification

If completions are present, clauses retain the resolved literals.

((P 1) (Q 1 2 )) resolves with (P (a))

to give

((MARK (P (a)) (Q(a) 2))

MARK is a one argument "predicate" taking the old predicate as its argument. (Many places in the code attempt to bypass marked literals.) If a marked literal is either at the left of a clause or next to another marked literal, its completion is executed and the literal deleted.

```
*                                                                          1
*                                                                          2
**           DILEMMA MASTER LISTING                                        3
*                                                                          4
*                                                                          5
*                                                                          6
** SYSTEM MACROS DEBUGGING TOOLS AND RELATED ODDS AND ENDS                  7
**                                                                         8
** IF DEBUG IS TRUE WE PRINT WHENEVER A DETAIL STATEMENT IS PRESENT         9
DETAIL(X1) MEANS IF DEBUG THEN PRINT(=X1,X1);                             10
SETPROP(X1,X2, X3) MEANS SETPROPY(X2,X1,X3);                              11
SUBCOPY(X1,X2) MEANS SUBINPLA(COPY(X1), X2);                              12
=X1=X2 MEANS BREAKUP(=X1,X2);                                             13
ATOM(X1) MEANS ¬PAIRQ(X1);                                                14
FALSE = NIL;                                                              15
**                                                                        16
**      MACROS TO GET PARTS OF CLAUSES AND LITERALS                       17
**                                                                        18
**    EACH CLAUSE IS REPRESENTED BY A 8 ELEMENT ARRAY                     19
** 1 CLAUSE ID , *0XX FOR GENERATED NAMES (INPUT CLAUSES ) VIA A PROVE     20
**        NUMBERS FOR RESOLVENTS                                          21
** 2 PTR TO NEG PARENT                                                    22
** 3 PTR TO LITERALS                                                      23
** 4 LENGTH IN LITERALS                                                   24
**           ONLY NEGATIVE ATOMS ARE COUNTED                             25
** 5 MIXED OR POSITIVE PARENT                                            26
** 6 EXTRA BINDINGS                                                      27
** 7 DEPTH OF CLAUSE (0 IS AN INPUT CLAUSE)                              28
** 8 MERIT OF CLAUSE                                                     29
**                                                                        30
NAMEOF(X1) MEANS X1[1] ;                                                  31
PREDSYM(X1) MEANS HD X1;                                                  32
ARGLIST(X1) MEANS TL X1;                                                  33
FNSYM(X1) MEANS PREDSYM(X1);                                              34
INPUTCL(X1) MEANS IDQ(X1[1]);                                             35
NEGPAR(X1) MEANS X1[2];                                                   36
LITLIST(X1) MEANS X1[3];                                                  37
CLENGTH(X1) MEANS X1[4];                                                  38
MIXEDPAR(X1) MEANS X1[5];                                                 39
BINDINGS(X1) MEANS X1[6];                                                 40
DEPTHVAL(X1) MEANS X1[7];                                                 41
MERITVAL(X1) MEANS X1[8];                                                 42
**                                                                        43
**             GENERAL UTILITY PROGRAMS                                   44
**              COPIES OF BALM UTILITIES WITH CHANGES                     45
**                                                                        46
** APPEND IS USED TO PUT ONE LIST ON THE BACK OF A SECOND                 47
** APPEND WILL CHANGE ITS FIRST ARGUMENT                                 48
APPEND=  PROC(LIST1,LIST2),                                               49
     BEGIN(B),                                                            50
        IF ¬LIST1 THEN RETURN LIST2,                                      51
        IF ¬LIST2 THEN RETURN LIST1,                                      52
        B= LIST1, WHILE TL LIST1 REPEAT LIST1= TL LIST1,                  53
        TL LIST1= LIST2, RETURN B                                         54
```

```
      END                                                                    55
   END;                                                                      56
RENAME=  PROC(CL,ID),                                                        57
      BEGIN(NEWNAMES),                                                       58
         NEWNAMES=NIL,RETURN MAPX(CL,                                        59
         NAMELIT)                                                            60
      END                                                                    61
   END;                                                                      62
* NAMELIT RENAMES THE VARIABLES IN A LITERAL TO NUMBERS THE LEFT MOST        63
* VARIABLE GETS THE VALUE OF ID , ID GETS CHANGED                           64
NAMELIT= PROC(LIT),                                                          65
      BEGIN(S),                                                              66
         IF ATOM(LIT) THEN DO                                                67
            IF ¬(S=LOOKUP(LIT,NEWNAMES) ) THEN NEWNAMES= (LIT:(S=ID=ID+1     68
            )):NEWNAMES,                                                     69
            RETURN S                                                         70
         END , TL LIT= MAPX(TL LIT,NAMELIT) ,RETURN LIT                      71
      END                                                                    72
   END;                                                                      73
LOOKUP=  PROC(ITEM,LIST),                                                    74
      BEGIN(),                                                               75
         WHILE LIST REPEAT DO                                                76
            IF EQUAL(HD HD LIST, ITEM) THEN RETURN TL HD LIST, LIST= TL      77
            LIST                                                             78
         END,                                                                79
         RETURN NIL                                                          80
      END                                                                    81
   END;                                                                      82
SUB1=    PROC(X,B),                                                          83
      BEGIN(Y),                                                              84
** APPLY BINDING LIST B TO X                                                 85
         IF ¬PAIRQ(X) THEN RETURN SUBINPLA(X, B),                            86
         Y=X,                                                                87
         WHILE X REPEAT DO                                                   88
            HD X = SUBINPLA(HD X , B), X= TL X                               89
         END,                                                                90
         RETURN Y                                                            91
      END                                                                    92
   END;                                                                      93
MAKNAME= PROC(CL),                                                           94
      BEGIN( NAMES,XVARS),                                                   95
         NAMES=NIL,XVARS=ZVARIABL,CL                                         96
         =MAPX(CL,NAME1)                                                     97
      END                                                                    98
   END;                                                                      99
DELETE=  PROC(ITEM,LIST),                                                   100
      BEGIN(B,E),                                                           101
         B=NIL:LIST,                                                        102
         LIST= B,                                                           103
         WHILE TL LIST REPEAT DO                                            104
            IF EQUAL(ITEM, HD TL LIST) THEN TL LIST= TL TL LIST ELSE        105
            LIST = TL LIST                                                  106
         END, RETURN TL B                                                   107
      END                                                                   108
```

```
  END;                                                                    109
MAPR=    PROC(ALIST,PRO),                                               110
     BEGIN(L),                                                          111
        L= ALIST, WHILE L REPEAT DO                                     112
           HD L= PRO(HD L), L= TL L                                     113
        END, RETURN ALIST                                               114
     END                                                                115
  END;                                                                  116
PRINTHIS=PROC(CL),                                                      117
     BEGIN(J),                                                          118
        J=NIL,                                                          119
        IF SAVER THEN DO                                                120
           PRINT(≠NULL CLAUSE ***≠), RETURN ()                          121
        END ,                                                           122
        WHILE CL REPEAT DO                                              123
           PRINT(NAMEOF(CL),LITLIST(CL),J), J= MIXEDPAR(CL), CL= NEGPAR 124
           (CL)                                                         125
           ,J= NAMEOF(J)                                                126
        END                                                             127
     END                                                                128
  END;                                                                  129
MAX1=    PROC(A,B),                                                     130
     IF A GT B THEN A ELSE B                                            131
  END;                                                                  132
CSTK=MAKVECTO(201);                                                     133
OLDCOPY= COPY;                                                          134
COPX=    PROC(Y),                                                       135
     BEGIN(I,K,X),                                                      136
        IF Y EQ NIL THEN RETURN Y,                                      137
        IF INTQ(Y) THEN RETURN Y,                                       138
        IF IDQ(Y) THEN RETURN Y,                                        139
        IF ¬PAIRQ(Y) THEN DO                                            140
           PRINT(≠COPY PROBLEMS≠, Y),RETURN OLDCOPY(Y)                  141
        END,                                                            142
        IF ¬PAIRQ(Y) THEN RETURN Y,                                     143
        K=1,I=3,CSTK[1]=HD Y, CSTK[2]= TL Y,                            144
        WHILE(K LT I) REPEAT DO                                         145
           IF PAIRQ(X= CSTK[K]) THEN DO                                 146
              CSTK[I]=K,CSTK[I+1]=HD X, CSTK[I+2]= TL X,                147
              I= I+3                                                    148
           END,                                                         149
           IF PAIRQ(X= CSTK[K=(K+1)])THEN DO                            150
              CSTK[I]=K,CSTK[I+1]=HD X, CSTK[I+2]=TL X, I= I+3          151
           END,                                                         152
           IF I GT 198 THEN DO                                          153
              COPYFAIL= COPYFAIL+1,                                     154
              K=X=NIL:NIL,                                              155
              I= Y, WHILE I REPEAT DO                                   156
                 X= TL X= COPY(HD I): NIL , I= TL I                     157
              END,RETURN TL K                                           158
           END,                                                         159
           K= K+2                                                       160
        END,                                                            161
        FOR K=(I-1,5,-3) REPEAT                                         162
```

```
        CSTK[CSTK[K-2]]=CSTK[K-1]:CSTK[K],                                  163
        RETURN CSTK[1]:CSTK[2]                                              164
     END                                                                    165
  END;                                                                      166
SUBINPLA=PROC(LIT,UNIF),                                                    167
     BEGIN(U,L2),                                                           168
        IF ¬LIT THEN RETURN NIL,                                            169
        WHILE ¬(U.UNIF) = UNIF REPEAT LIT= SUBER(LIT,HD U , TL U)           170
        , RETURN (LIT)                                                      171
     END                                                                    172
  END;                                                                      173
SUBER=   PROC(LIST,ITEM,REPLACE),                                           174
     BEGIN(L2),                                                             175
        IF EQUAL( LIST, ITEM) THEN RETURN (REPLACE),                        176
        IF ATOM(LIST) THEN RETURN LIST,                                     177
        L2= TL LIST, WHILE L2 REPEAT DO                                     178
           HD L2= SUBER(HD L2,ITEM , REPLACE), L2= TL L2                    179
        END,                                                                180
        RETURN (LIST)                                                       181
     END                                                                    182
  END;                                                                      183
**                                                                          184
**         INPUT PARSER                                                     185
**                                                                          186
**                                                                          187
* NAME1 IS A VARIATION OF NAMELIT. HOWEVER, VARIABLES ARE                   188
* REPLACED BY ELEMENTS OF A LIST RATHER THEN NUMBERS                        189
NAME1=   PROC(LIT),                                                         190
     BEGIN( S ),                                                            191
        IF ¬LIT THEN RETURN NIL,                                            192
        IF ATOM(LIT) THEN DO                                                193
           S= LOOKUP(LIT,NAMES), IF ¬S THEN DO                              194
              IF ¬XVARS THEN DO                                             195
                 PRINT(=NAME1,=XVARS,=EMPTY),RETAIN=FALSE,XVARS=            196
                 XVARIABL                                                   197
              END,                                                          198
              S= HD XVARS, XVARS= TL XVARS, NAMES=(LIT:S) : NAMES           199
           END,                                                             200
           RETURN S                                                         201
        END,                                                                202
        RETURN HD LIT: MAPX(TL LIT, NAME1)                                  203
     END                                                                    204
  END;                                                                      205
MEMB=    PROC(ITEM,ALIST),                                                  206
     BEGIN(),                                                               207
        WHILE PAIRQ(ALIST) REPEAT DO                                        208
           IF EQUAL(ITEM,HD ALIST) THEN RETURN TRUE, ALIST= TL ALIST        209
        END,                                                                210
        RETURN ALIST EQ ITEM                                                211
     END                                                                    212
  END;                                                                      213
REMCO=   PROC(X),                                                           214
     BEGIN(XX,Y),                                                           215
        XX=X,                                                               216
```

```
      IF ATOM(XX) THEN DO                                                    217
         IF MEMBER(XX,CONSTANT                                               218
            ) THEN RETURN LIST(XX) ELSE RETURN XX                            219
         END,                                                                220
         IF HD XX EQ = ^ THEN DO                                             221
            ANDSW= TRUE, HD XX= =,                                           222
         END,                                                                223
         IF HD XX EQ =, THEN DO                                              224
            IF HD HD TL TL XX EQ = ^ THEN DO                                 225
               ANDSW = TRUE, HD HD TL TL XX = =,                             226
            END,                                                             227
            IF HD HD TL TL XX EQ =, THEN                                     228
            RETURN(REMCO(HD TL XX) : REMCO(HD TL TL XX) )                    229
            ELSE RETURN (REMCO(HD TL XX): REMCO(HD TL TL XX): NIL)           230
         END,                                                                231
         WHILE XX REPEAT DO                                                  232
            IF HD HD XX EQ = ^ THEN DO                                       233
               HD HD XX= =, ,ANDSW= TRUE                                     234
            END,                                                             235
            IF HD HD XX EQ =, THEN DO                                        236
               Y= HD XX, HD XX= HD TL Y, TL XX=(HD TL TL Y): TL XX            237
            END,                                                             238
            HD XX = REMCO(HD XX), XX= TL XX                                  239
         END, RETURN X                                                       240
      END                                                                    241
   END;                                                                      242
REMCO1= REMCO;                                                               243
CONST = PROC(X),                                                             244
   BEGIN(),                                                                  245
      IF ATOM(X) THEN CONSTANT                                               246
         =X:CONSTANT                                                         247
            ELSE                                                             248
            CONSTANT                                                         249
               =APPEND(REMCO(X),CONSTANT                                     250
                  )                                                          251
               END                                                           252
            END;                                                             253
REMOC1= REMCO;                                                               254
DIFFEREN (X1,X2) MEANS NOTEQLIS= (REMCO(=X1) : REMCO(=X2)): NOTEQLIS;        255
BRACKET(=CONSTANT                                                            256
   ,100,=CONSTANT                                                            257
      );                                                                     258
CONSTANT                                                                     259
   X1                                                                        260
END MEANS CONST(=X1);                                                        261
INFIX(=ASSERT,1451,1450,=ASSERT);                                            262
X1 ASSERT X2 MEANS MIXEDCL=GENCLAUS(=X1,REMOC1(=X2),NIL):MIXEDCL;            263
INFIX(=THEOREM,1501,1500,=THM);                                              264
INFIX(=FROM,1400,1400,=FRM);                                                 265
INFIX(=COER,1451,1450,=COER);                                                266
UNARY(=TYPE,100,=TY);                                                        267
TYPE X1 = X2 MEANS SETPROP(=X1,=TYPE ,X2);                                   268
FIXCO=   PROC(X),                                                            269
   BEGIN(Y),                                                                 270
```

```
        ANDSW= FALSE,                                                    271
        Y=REMCO(X), IF ATOM(Y) THEN RETURN Y,                            272
        IF ¬ANDSW THEN RETURN LIST(Y),                                   273
        RETURN Y                                                         274
      END                                                                275
  END;                                                                   276
INFIX(= ^  , 1401,1401, = ^ );                                           277
X1 THEOREM X2 FROM X3 MEANS MIXEDCL=GENCLAUS(=X1,REMCO(=X2),FIXCO        278
(=X3)): MIXEDCL ;                                                        279
X1 COER X2, X3 MEANS GENCOER(=X1,REMCO1(=X2),REMCO1(=X3),NIL);           280
X1 COER X2,X3,X4 MEANS GENCOER(=X1,REMCO1(=X2),REMCO1(=X3),FIXCO(=X4));  281
PROVE(X1) MEANS OTTER(=X1);                                              282
INFIX(=COMPLETE,1401,1400,=CPL);                                         283
INFIX(=CPROC,1401,1400,=CPRPC);                                          284
INFIX(=PART,1401,1400,=PART);                                            285
INFIX(=PARTEVAL,1401,1400,=PE);                                          286
X1 COMPLETE X2 MEANS DO                                                  287
   SETPROP(=X2,=CMPL,X1), COMPLET = TRUE                                 288
END;                                                                     289
X1 CPROC X2 MEANS DO                                                     290
   SETPROP(=X2,=CPROC,X1)                                                291
END;                                                                     292
X1 PART 1 = X2 MEANS DO                                                  293
   PEFNS= TRUE, SETPROP(=X1,=PE1,X2)                                     294
END;                                                                     295
X1 PART 2 = X2 MEANS DO                                                  296
   PEFNS= TRUE, SETPROP(=X1,=PE2,X2)                                     297
END;                                                                     298
X1 PART 3 = X2 MEANS DO                                                  299
   PEFNS= TRUE, SETPROP(=X1,=PE3,X)                                      300
END;                                                                     301
X1 PART 4 = X2 MEANS DO                                                  302
   PEFNS= TRUE, SETPROP(=X1,=PE4,X2)                                     303
END;                                                                     304
X1 PART 5 = X2 MEANS DO                                                  305
   PEFNS= TRUE, SETPROP(=X1,=PE5,X2)                                     306
END;                                                                     307
X1 PART 6 = X2 MEANS DO                                                  308
   PEFNS= TRUE, SETPROP(=X1,=PE6,X2)                                     309
END;                                                                     310
X1 PARTEVAL X2 MEANS DO                                                  311
   PEFNS= TRUE, SETPROP(=X1,=PE,X2)                                      312
END;                                                                     313
*                                                                        314
*       CODE GENERATORS                                                  315
*                                                                        316
* GENCLAUSE GETS NAME OF CLAUSE, POSITIVE LITERAL, AND A LIST OF         317
* NEGATIVE LITERALS AS ITS THREE ARGUMENTS.                              318
* IT CREATES A VECTOR RESULT                                             319
* NOTE THAT PE DONT GET EXECUTED HERE NOR DO COMPLETIONS OF POSITIVE     320
* UNITS                                                                  321
* GENCLAUSE DOES NOT RENAME CLAUSES TO XVARS, NOR DOES IT CHECK          322
* IF A CLAUSE SUBSUMES OR IS A DUPLICATE OF AN EXISTING CLAUSE           323
* PROC GENNEG IS THE RUNTIME GENERATOR . IT IS IN THE POST UNIFICATION    324
```

```
*        SECTION.                                                              325
GENCOER= PROC(NAME,A,B,C),                                                     326
     BEGIN(XVARS, NAMES),                                                      327
        NAMES= NIL,XVARS=ZVARIABL,                                             328
        COERS= NAME:COERS,                                                     329
        SETPROP(NAME,=PAIR,NAME1(A):NAME1(B))                                  330
        , VALUE NAME= VECTOR(NAME,NIL,MAPX(C,NAME1),LENGTH(C),NIL,NIL,0        331
        ,0),                                                                   332
        V= COPY(ZVARIABL), WHILE XVARS REPEAT DO                               333
           V= DELETE(HD XVARS,V), XVARS= TL XVARS                              334
        END,                                                                   335
        DETAIL(V),                                                             336
        SETPROP(NAME,=VARS,V), RETURN VALUE NAME                               337
     END                                                                       338
  END;                                                                         339
GENCLAUS=PROC(NAME,POS,NEG),                                                   340
     BEGIN(P),                                                                 341
        VALUE NAME = VECTOR(NAME,NIL,NIL,LENGTH(NEG) ,NIL,NIL,0,0),            342
        LITLIST(VALUE NAME) = NEG,                                             343
        SETPROP(NAME,=POSLIT,POS),                                             344
        MERITVAL(VALUE NAME)= COMPLEX(NEG) ,                                   345
        IF POS THEN DO                                                         346
           P= PREDCHAI, WHILE P REPEAT DO                                      347
              IF HD POS EQ HD HD P THEN DO                                     348
                 TL HD P=(VALUE NAME):(TL HD P),                               349
                 RETURN VALUE NAME                                             350
              END, P= TL P                                                     351
           END,                                                                352
           PREDCHAI= ((HD POS):((VALUE NAME):NIL)):PREDCHAI                    353
        END,                                                                   354
        RETURN VALUE NAME                                                      355
     END                                                                       356
  END;                                                                         357
*                                                                              358
*          UNIFICATION                                                         359
*                                                                              360
ZVARIABL= =(Z1 Z2 Z3 Z4 Z5 Z6 Z7 Z8 Z9 Z10 Z11 Z12);                           361
SETBIND(X1,X2) MEANS                                                           362
((( X1:X2):NIL):NIL) ;                                                         363
INFIX(=ISNOTIN,1401,1200,=NOTIN);                                              364
EMP1= LIST(NIL);                                                               365
EMPTY = LIST(LIST(NIL));                                                       366
** NOTIN IS A UNIFICATION PRIMATIVE  VAR NOTIN EXP                            367
NOTIN=   PROC(VAR,ALIST),                                                      368
     BEGIN(),                                                                  369
        IF ¬PAIRQ(ALIST) THEN RETURN (¬ EQUAL(VAR,ALIST)),                     370
        ALIST= TL ALIST,                                                       371
        WHILE ALIST REPEAT DO                                                  372
           IF ¬NOTIN(VAR, HD ALIST) THEN RETURN FALSE,                         373
           ALIST= TL ALIST                                                     374
        END , RETURN TRUE                                                      375
     END                                                                       376
  END;                                                                         377
*  THE PAIR OF A COERCION ARE COPIES OF THE TWO EQUIV EXPRESSIONS              378
```

```
* WHERE VARIABLES HAVE BEEN REPLACED BY NUMBERS                                  379
PUSHUNIF=PROC(OLD,NEW),                                                         380
     BEGIN(U1,U2,L1,L2,U3,U,P,L4),                                              381
** OLD AND NEW ARE TWO UNIFER LISTS WHICH ARE TO BE COMBINED AND UPDATE         382
        IF EQUAL ( OLD , EMP1) THEN RETURN NEW,                                 383
        IF EQUAL( EMP1,NEW) THEN RETURN OLD,                                    384
        =(U1.L1)=OLD,=(U2.L2)=NEW,                                              385
        L1= SUB1(L1,U1), L1= SUB1(L1,U2),                                       386
        L2= SUB1(L2,U2),                                                        387
        L4=L2,                                                                  388
        IF L2 THEN                                                              389
        DO                                                                      390
           WHILE TL L2 REPEAT L2= TL L2, TL L2= L1                              391
        END ELSE L4= L1,                                                        392
        U3 = U1,                                                                393
        WHILE =(P.U3)=U3 REPEAT TL P= SUBINPLA(TL P,U2),                        394
        U3 = U1,                                                                395
        IF U3 THEN DO                                                           396
           WHILE TL U3 REPEAT U3= TL U3, TL U3=U2                               397
        END ELSE U1=U2,                                                         398
        RETURN U1:L4                                                            399
     END                                                                        400
  END;                                                                          401
MGU1=     PROC(P,N,A,B,OUT,DEPTH,COR,BINDS,NAME),                               402
     BEGIN(M1,M,P1,U,M2,M3, MHOLD),                                             403
        M=M2=NIL,                                                               404
        IF BVARS THEN DO                                                        405
           IF ATOM(P) AND ATOM(N) THEN RETURN UOUT                              406
        END,                                                                    407
        UNIFDEPT=UNIFDEPT+1,                                                    408
        IF BVARS THEN DO                                                        409
           IF ATOM(P) AND ATOM(A) THEN M= LIST(LIST(A:P):NIL)                   410
        END , IF ¬M THEN                                                        411
        M= (IF U1 THEN EMPTY ELSE MGUSET(P,A,DEPTH,ADDLITS- LENGTH(COR)         412
        ,NAME)),                                                                413
        WHILE =(M1.M) = M REPEAT DO                                             414
           M2= NIL,                                                             415
           IF BVARS THEN DO                                                     416
              IF ATOM(N) AND ATOM(B) THEN M2= LIST(LIST(B:N):NIL)               417
           END ,IF ¬M2 THEN                                                     418
           M2= (IF U2 THEN EMPTY ELSE MGUSET(SUBCOPY(N,HD M1),SUBCOPY(B         419
           ,HD M1),                                                             420
           DEPTH,ADDLITS- LENGTH(COR)- LENGTH(TL M1),NAME)),                    421
           WHILE(=(M3.M2)=M2) REPEAT DO                                         422
              UOUT= PUSHUNIF(PUSHUNIF(COPY(M1),M3),(COPY(BINDS):COPY(           423
              COR))):UOUT                                                       424
           END                                                                  425
        END,                                                                    426
        UNIFDEPT=UNIFDEPT-1,                                                    427
        RETURN UOUT                                                             428
     END                                                                        429
  END;                                                                          430
POSSIBLE=PROC(A, N , FLAG),                                                     431
     BEGIN(),                                                                   432
```

```
        VALUE FLAG = NIL,                                                    433
        IF ATOM(A) THEN RETURN TRUE,                                         434
        IF ATOM(N) THEN RETURN TRUE ,                                        435
        IF HD A EQ HD N THEN RETURN TRUE, RETURN FALSE                       436
      END                                                                    437
   END;                                                                      438
MGUSET=  PROC(N,P,DEPTH,ADDLITS,DONT),                                       439
      BEGIN(NT,PT,P1,N1, UNIFS, UOUT,E1, U2,B, U1,A,E,M, LCOR, BIND,         440
        BINDS,                                                               441
        U,J,J2,NAME),                                                        442
        IF EQUAL(N,P) THEN RETURN (EMPTY),                                   443
        N1= NOTEQLIS,                                                        444
        WHILE N1 REPEAT DO                                                   445
           NT= HD N1, N1= TL N1,                                             446
           IF EQUAL(HD NT,P) AND EQUAL(TL NT,N) THEN RETURN =NOGOOD,         447
           IF EQUAL(HD NT,N) AND EQUAL(TL NT,P) THEN RETURN =NOGOOD          448
        END,                                                                 449
        IF DEPTH GT MAXCOERS THEN RETURN =NOGOOD,                            450
        UOUT=UNIFS=NIL,                                                      451
        IF ¬BYPASS THEN DO                                                   452
           IF ATOM(P) THEN DO                                                453
              IF P ISNOTIN N THEN UNIFS= LIST(SETBIND(P,N))                  454
           END                                                               455
           ELSEIF ATOM(N) THEN DO                                            456
              IF N ISNOTIN P THEN UNIFS= LIST(SETBIND(N,P))                  457
           END,                                                              458
           UOUT=UNIFS,                                                       459
           IF PAIRQ(N) AND PAIRQ(P) AND(HD N EQ HD P) THEN DO                460
              UNIFS=EMPTY,                                                   461
              NT = TL N, PT= TL P,                                           462
              WHILE NT REPEAT DO                                             463
                 N1= HD NT, NT= TL NT,                                       464
                 P1= HD PT, PT= TL PT, UOUT= NIL,                            465
                 WHILE UNIFS REPEAT DO                                       466
                    U= HD UNIFS, UNIFS= TL UNIFS,                            467
                    IF ADDLITS LT LENGTH(TL U) THEN M= =NOGOOD ELSE DO       468
                       A= COPY(N1),B=COPY(P1),                               469
                       UNIFDEPT=UNIFDEPT+1,                                  470
                       M=MGUSET(SUBINPLA(A,HD U), SUBINPLA(B,HD U),          471
                       DEPTH,                                                472
                       ADDLITS-LENGTH(TL U),NIL),                            473
                       UNIFDEPT=UNIFDEPT-1                                   474
                    END,                                                     475
                    WHILE PAIRQ(M) REPEAT DO                                 476
                       E= HD M, M= TL M,                                     477
                       UOUT=PUSHUNIF(COPY(U),E):UOUT                         478
                    END                                                      479
                 END, UNIFS= UOUT                                            480
              END                                                            481
           END                                                               482
        END, BYPASS=NIL,                                                     483
        E1=(ATOM(N) OR ATOM(P)),                                             484
        IF ¬E1 THEN E1= (HD N EQ HD P),                                      485
        IF ¬E1 THEN E1= ALWAYS,                                              486
```

```
        IF E1 THEN DO                                                    487
           IF (DEPTH LT MAXCOERS) THEN DO                                488
              E= COERS,                                                  489
              WHILE E REPEAT DO                                          490
                 E1= HD E, E= TL E,                                      491
                 IF E1 NE DONT THEN DO                                   492
                    NAME= E1,                                            493
                    IF CLENGTH(VALUE E1) LE ADDLITS THEN DO              494
                       J2=GETPROP(E1,=CPROC),                            495
                       IF COPYPAIR(E1,N,P) THEN DO                       496
                          M= BINDS,                                      497
                          IF(¬ATOM(A) AND ¬ATOM(B)) OR (UNIFDEPT NE 0)   498
                          THEN DO                                        499
                             BINDS= NIL,                                 500
                             E1= LCOR,                                   501
                             IF CODEQ(J2) THEN J=J2(N,P,UOUT) ELSE J=    502
                             TRUE,                                       503
                             IF J THEN DO                                504
                                N1= 0, IF J NE TRUE THEN N1=(MAXCOERS -  505
                                J ), N1= MAX1(N1,DEPTH+1),               506
                                IF POSSIBLE(A,N,=U1) AND POSSIBLE(B,P,=  507
                                U2) THEN DO                              508
                                   UOUT=MGU1(N,P,A,B,OUT,N1,COPY(E1),    509
                                   BINDS,NAME)                           510
                                END                                      511
                             END,                                        512
                             IF (¬ONEWAY) OR (¬UOUT) THEN DO             513
                                IF CODEQ(J2) THEN J=J2(P,N,UOUT) ELSE J  514
                                = TRUE,                                  515
                                IF J THEN DO                             516
                                   N1= 0, IF J NE TRUE THEN N1=(         517
                                   MAXCOERS -J ), N1= MAX1(N1,DEPTH+1),  518
                                   IF POSSIBLE(B,N,=U2) AND POSSIBLE(A,  519
                                   P,=U1) THEN DO                        520
                                      UOUT=MGU1(P,N,A,B,OUT,N1,E1,BINDS  521
                                      ,NAME)                             522
                                   END                                   523
                                END                                      524
                             END                                         525
                          END                                            526
                       END                                               527
                    END                                                  528
                 END                                                     529
              END                                                        530
           END                                                           531
        END,                                                             532
        IF ¬UOUT THEN UOUT = =NOGOOD,                                    533
        RETURN UOUT                                                      534
     END                                                                 535
  END ;                                                                  536
COPYPAIR=PROC(COR,S,T),                                                  537
     BEGIN(P,V,V1),                                                      538
        P=GETPROP(COR,=PAIR),                                            539
        IF¬POSSIBLE(HD P,S,=V) THEN DO                                   540
```

```
            IF¬POSSIBLE(TL P, S , =V) THEN RETURN FALSE,                    541
            IF ¬POSSIBLE(HD P , T ,=V) THEN RETURN FALSE                    542
         END,                                                               543
         IF ¬POSSIBLE(TL P, T ,=V) THEN RETURN FALSE,                       544
         P= COPY(GETPROP(COR,=PAIR)),V= GETPROP(COR,=VARS),                 545
         BIND= NIL, WHILE =(V1.V) = V REPEAT                                546
         BIND=(V1:(CORID=CORID+1)):BIND, LCOR= SUB1(COPY(LITLIST(VALUE      547
         COR))                                                              548
         , BIND) ,                                                          549
         A=SUBINPLA(COPY(HD P),BIND),B=SUBINPLA(COPY(TL P),BIND),           550
         RETURN TRUE                                                        551
      END                                                                   552
   END;                                                                     553
VARSNUM= PROC(LIT),                                                         554
      BEGIN(S,L2),                                                          555
         S=0,L2=TL LIT, WHILE L2 REPEAT DO                                  556
            IF ATOM(HD L2) THEN S= S+1 ELSE S= S+VARSNUM(HD L2),L2= TL      557
            L2                                                              558
         END,                                                               559
         RETURN S                                                           560
      END                                                                   561
   END;                                                                     562
*                                                                           563
*          POST UNIFICATION PROCESSING                                      564
*                                                                           565
EMPTYCLA= PROC(X),                                                          566
      CLENGTH(X) EQ 0                                                       567
   END;                                                                     568
*  SUBTRACT IS USED TO BUILD RESOLVENTS.                                    569
* IT BOTH APPLIES A UNIFIER AND REMOVES OCCURANCES OF THE LITERAL           570
SUBTRACT=PROC(LIST,LIT,UNIF),                                               571
      BEGIN(L , L1),                                                        572
         LIT= SUBINPLA(LIT,UNIF),                                           573
         L= NIL,                                                            574
         WHILE LIST REPEAT DO                                               575
            L1= SUBCOPY(HD LIST,UNIF),                                      576
            IF ¬EQUAL(L1,LIT) THEN L= L1:L,                                 577
            LIST= TL LIST                                                   578
         END, RETURN L                                                      579
      END                                                                   580
   END;                                                                     581
FAKECL=VECTOR(=FAKECL,NIL,0,NIL,NIL,0,-100);                                582
CPLEX=   PROC(CL),                                                          583
      MERITVAL(CL) + DEPTHVAL(CL) / MAXCPLEX                                584
   END;                                                                     585
NEST=    PROC(CL,D),                                                        586
      BEGIN(M,M2),                                                          587
         IF ATOM(CL) THEN RETURN D,                                         588
         M=D,                                                               589
         WHILE CL REPEAT DO                                                 590
            M2= NEST(HD CL, D+1),                                           591
            IF M2 GT M THEN M= M2, CL= TL CL                                592
         END,                                                               593
         RETURN M                                                           594
```

```
      END                                                                    595
   END;                                                                      596
CLMEMBER=PROC(CL,LIST),                                                      597
      BEGIN(M,W),                                                            598
         M=MERITVAL(CL),                                                     599
         CL=LITLIST(CL),                                                     600
         WHILE LIST REPEAT DO                                                601
            W= HD LIST, IF MERITVAL(W) EQ M THEN DO                          602
               IF SAMECL(CL,W) THEN DO                                       603
                  IF CLAUSEPR THEN                                           604
                  PRINT(≠DUPLICATE OF≠,NAMEOF(W)), RETURN TRUE               605
               END                                                           606
            END,                                                             607
            LIST= TL LIST                                                    608
         END, RETURN NIL                                                     609
      END                                                                    610
   END;                                                                      611
CLEQ=    PROC(LIT1,LIT2),                                                    612
      BEGIN(),                                                               613
         IF EQUAL(LIT1,LIT2) THEN RETURN TRUE,                               614
         IF (( PREDSYM(LIT1) EQ EQ1) OR (PREDSYM(LIT1) EQ EQ2 ))             615
         AND ((PREDSYM(LIT2) EQ EQ1) OR (PREDSYM(LIT2) EQ EQ2 ))             616
         THEN                                                                617
         DO                                                                  618
            IF EQUAL(TL LIT1,TL LIT2) THEN RETURN TRUE,                      619
            IF ¬ EQUAL( HD TL LIT1,HD TL TL LIT2) THEN RETURN FALSE,         620
            RETURN EQUAL(HD TL LIT2, HD TL TL LIT1)                          621
         END, RETURN FALSE                                                   622
      END                                                                    623
   END;                                                                      624
SAMECL=  PROC(LIT,CPR),                                                      625
      BEGIN(CPRLIT),                                                         626
         CPRLIT=LITLIST(CPR),                                                627
         RETURN SEQUAL(LIT,CPRLIT)                                           628
      END                                                                    629
   END;                                                                      630
MOVE=    PROC(LIST),                                                         631
      BEGIN(),                                                               632
         WHILE LIST REPEAT DO                                                633
            IF HD HD LIST NE =MARK THEN RETURN LIST, LIST= TL LIST            634
         END, RETURN NIL                                                     635
      END                                                                    636
   END;                                                                      637
GENNEG=  PROC(NCTX,MCTX,NEGLIT,UNIFER,NEGLITS),                              638
      BEGIN(CL,PLIT,LITS),                                                   639
         NEGLITS= COPY(NEGLITS),                                             640
         PLIT=COPY(LITLIST(MCTX)),                                           641
         CURCONTE = CL= VECTOR (CLAUSEID=CLAUSEID+1,NCTX,NIL,LENGTH(PLIT     642
         )                                                                   643
         + CLENGTH(NCTX) -1, MCTX,NIL,DEPTHVAL(NCTX)+1,MERITVAL(NCTX)),      644
         IF SAVER THEN MIXEDPAR(CURCONTE)= NIL,                              645
         IF SAVER THEN NEGPAR(CURCONTE)= NIL,                                646
         LITS= APPEND(TL UNIFER,APPEND(PLIT,NEGLITS)),                       647
         DETAIL(LITS),                                                       648
```

```
      PLIT=LITS, WHILE PLIT REPEAT DO                                          649
         HD PLIT = SUBINPLA(HD PLIT,HD UNIFER), PLIT = TL PLIT                 650
      END,                                                                     651
      DETAIL(LITS),                                                            652
      IF COMPLET THEN                                                          653
      LITS=MARK(LITS,SUBINPLA(NEGLIT,HD UNIFER)) ELSE LITS= DELETE(            654
      SUBINPLA(NEGLIT,HD UNIFER),LITS),                                        655
      DETAIL(LITS),                                                            656
      MERITVAL(CL)= COMPLEX(LITS),                                             657
      IF PEFNS THEN DO                                                         658
         NEWBINDS= NIL, NAMES=NIL,XVARS=XVARIABL,LITS=MAPX(LITS,NAME1          659
         ),                                                                    660
         IF RETAIN THEN                                                        661
         LITS= EVALUATE(LITS)                                                  662
      END,                                                                     663
      DETAIL(LITS),                                                            664
      IF COMPLET THEN                                                          665
      TESTCOMP(LITS),                                                          666
      IF COMPLET THEN                                                          667
      LITS= MERGE(LITS),                                                       668
      LITS= RENAME(RENAME(LITS,CORID+2),0),                                    669
      DETAIL(LITS),                                                            670
      LITLIST(CL)=LITS,                                                        671
      CLENGTH(CL)= LENGTH(LITS),                                               672
      RETURN CL                                                                673
   END                                                                         674
 END;                                                                          675
SEQUAL=  PROC(LITS,LITS2),                                                     676
   BEGIN(),                                                                    677
      LITS=MOVE(LITS), LITS2=MOVE(LITS2),                                      678
      WHILE LITS REPEAT DO                                                     679
         IF ¬CLEQ(HD LITS , HD LITS2) THEN RETURN FALSE,                       680
         LITS=MOVE(TL LITS),LITS2=MOVE(TL LITS2),                              681
         IF ¬ LITS THEN RETURN ¬LITS2                                          682
      END, RETURN FALSE                                                        683
   END                                                                         684
 END;                                                                          685
ADDNEGS= PROC(CL),                                                             686
   BEGIN(N,M,B),                                                               687
      N=FAKECL:NEGS,M=CPLEX(CL),B=N,                                           688
      RACCPT= RACCPT+1,                                                        689
      NUMSOLUT= NUMSOLUT+1,                                                    690
      WHILE TL N AND(CPLEX(HD TL N) LT M)REPEAT                                691
      N= TL N, TL N= CL: TL N, NEGS= TL B, RETURN()                            692
   END                                                                         693
 END;                                                                          694
DELATE=  PROC(CL,RETAIN),                                                      695
   BEGIN(),                                                                    696
      IF DEPTHVAL(CL) GT MAXLEVEL THEN DO                                      697
         RETAIN= FALSE,                                                        698
         DEEPCNTR= DEEPCNTR+1,                                                 699
         PRINT(≠DEPTH CUTOFF≠)                                                 700
      END                                                                      701
      ELSE IF NEST(LITLIST(CL),0) GT MAXNEST THEN DO                           702
```

```
            NSTR= NSTR+1,                                                    703
            PRINT(≠TOO DEEP≠), RETAIN= FALSE                                 704
         END                                                                 705
         ELSE IF MERITVAL(CL) GT MAXCPX THEN DO                              706
            CNTR= CNTR+1,                                                    707
            RETAIN = FALSE, PRINT(≠TOO COMPLEX≠)                             708
         END,                                                                709
         RETURN RETAIN                                                       710
      END                                                                    711
   END;                                                                      712
*                                                                            713
*              PARTIAL EVALUATION OF PREDICATES AND FUCTIONS                 714
*                                                                            715
CADJUST(X1) MEANS IF MEMBER(X1,CONSTANT                                      716
   ) THEN X1= LIST(X1);                                                      717
PENAMES = =(PE1 PE2 PE3 PE4 PE5 PE6);                                        718
* WHEN WE START TO PE A CLAUSE WE FIRST CHANGE ALL VARIABLES INTO             719
* MEMBERS OF THE LIST  VARIABLES. THIS LETS IDQ WORK                         720
*ISBOUND RETURNS TRUE IS ITS ARGUMENT HAS A BINDING ON THE SPECIAL            721
* BINDING LIST FOR THIS CLAUSE                                              722
XVARIABL= =(X1 X2 X3 X4 X5 X6 X7 X8 X9 X10 X11 X12);                         723
ISBOUND= PROC(VAR,CONTEXT),                                                  724
      BEGIN(),                                                               725
         CADJUST(VAR),                                                       726
         IF(BNDVAL=LOOKUP(VAR,NEWBINDS))THEN RETURN TRUE,                    727
         IF NUMARGS() EQ 1 THEN CONTEXT= CURCONTE,                           728
         RETURN ISBND1(VAR,CONTEXT)                                          729
      END                                                                    730
   END;                                                                      731
ISBND1=  PROC(VAR,CTX),                                                      732
      BEGIN(),                                                               733
* THE GLOBAL VARIABLE BNDVAL GETS THE BINDING                                734
         IF (BNDVAL= LOOKUP(VAR,BINDINGS(CTX))) THEN RETURN TRUE,            735
         IF INPUTCL(CTX) THEN RETURN FALSE,                                  736
         IF ISBND1(VAR,NEGPAR(CTX)) THEN RETURN TRUE                         737
      END                                                                    738
   END;                                                                      739
VARIABLE=PROC(ITEM),                                                         740
      ¬PAIRQ(ITEM) AND ¬MEMBER(ITEM,CONSTANT                                 741
         )                                                                   742
      END;                                                                   743
BIND=    PROC(EXP1,EXP2),                                                    744
      BEGIN(J),                                                              745
         CADJUST(EXP1),CADJUST(EXP2),                                        746
         IF ¬SAVER THEN                                                      747
         NEWBINDS=(EXP1 : EXP2) :NEWBINDS,                                   748
         MASTPRED=(SUBINPLA(MASTPRED,EXP1: EXP2 :NIL ))                      749
      END                                                                    750
   END;                                                                      751
PE=      PROC(PRED),                                                         752
      BEGIN(S,A,J),                                                          753
         S=PREDSYM(PRED), A=ARGLIST(PRED), MAPR(A,PEVAL1),                   754
         IF CODEQ(J=GETPROP(S,=PE)) THEN RETURN J(PRED)                      755
         ELSE RETURN PRED                                                    756
```

```
      END                                                                    757
   END;                                                                      758
PEVAL=   PROC(FN),                                                           759
      BEGIN(S,A,J),                                                          760
         S=FNSYM(FN),A=ARGLIST(FN), MAPR(A,PEVAL1),                          761
         IF CODEQ(J=GETPROP(S,=PE))THEN DO                                   762
            S=J(FN),                                                         763
            CADJUST(S), RETURN S                                             764
         END,                                                                765
         RETURN FN                                                           766
      END                                                                    767
   END;                                                                      768
PEVAL1=  PROC(X),                                                            769
      BEGIN(Y),                                                              770
         IF ATOM(X) THEN RETURN X, IF ¬TL X THEN RETURN X ,                  771
         Y=PEVAL(COPY(X)),                                                   772
         IF ¬EQUAL(X,Y) THEN BIND(X,Y) , RETURN Y                            773
      END                                                                    774
   END;                                                                      775
EVALUATE = PROC(CL),                                                         776
      BEGIN(B,NEWBINDS),                                                     777
         B= CL,                                                              778
         WHILE B REPEAT DO                                                   779
            NEWBINDS=NIL,                                                    780
            IF HD HD B NE =MARK THEN DO                                      781
               WORK=PARTEV(HD B),                                            782
               DETAIL(WORK),                                                 783
               IF WORK EQ TRUE THEN CL= DELETE(HD B, CL),                    784
               IF WORK EQ FALSE THEN DO                                      785
                  RETAIN= FALSE, PRINT(≠EVAL REJECTS≠),                      786
                  RETURN CL                                                  787
               END,                                                          788
               IF NEWBINDS THEN DO                                           789
                  S= NIL, SUBINPLA(NIL:CL, NEWBINDS),                        790
                  WHILE NEWBINDS REPEAT DO                                   791
                     IF ATOM(HD HD NEWBINDS) THEN DO                         792
                        IF ¬ MEMBER(HD HD NEWBINDS, XVARIABL) THEN DO        793
                           S= (HD NEWBINDS) : S                              794
                        END                                                  795
                     END,NEWBINDS= TL NEWBINDS                               796
                  END,                                                       797
                  WHILE S REPEAT DO                                          798
                     NEWBINDS= (HD S): NEWBINDS, S= TL S                     799
                  END,                                                       800
                  IF NEWBINDS THEN BINDINGS(CURCONTE) = NEWBINDS:            801
                  BINDINGS(CURCONTE)                                         802
               END                                                           803
            END,B= TL B                                                      804
         END, RETURN CL                                                      805
      END                                                                    806
   END;                                                                      807
CRACK=   PROC(PAT,EXP),                                                      808
      BEGIN(S),                                                              809
         IF IDQ(PAT) THEN DO                                                 810
```

```
            IF(CODEQ(S=GETPROP(PAT,=TYPE))) THEN DO                          811
               IF PAIRQ(EXP) AND ¬ARGLIST(EXP) THEN DO                       812
                  S= S(HD EXP),EXP= HD EXP                                   813
               END                                                           814
               ELSE S= S(EXP),                                               815
               IF ¬ S THEN RETURN FALSE                                      816
            END,                                                             817
            VALUE PAT = EXP, RETURN TRUE                                     818
         END,                                                                819
         IF ATOM(PAT) THEN RETURN EQUAL(PAT,EXP),                            820
         IF ATOM(EXP) THEN RETURN FALSE,                                     821
         IF CRACK(HD PAT,HD EXP) THEN RETURN CRACK(TL PAT, TL EXP),          822
         RETURN FALSE                                                        823
      END                                                                    824
   END;                                                                      825
PARTEV = PROC(PRED),                                                         826
      BEGIN(CONTEXT,MERIT,S,NFN,SA,DEPTH,J,V,MASTPRED),                      827
         MASTPRED=PRED,                                                      828
         CONTEXT=CURCONTE,MERIT=MERITVAL(CONTEXT), DEPTH=DEPTHVAL(           829
         CONTEXT),                                                           830
         S= PE(PRED),                                                        831
         MERITVAL(CONTEXT)=MERIT,                                            832
         IF ¬PAIRQ(S) THEN RETURN S,                                         833
         IF ¬PAIRQ(HD S) THEN RETURN S ,                                     834
         NFN= PENAMES, SA= ARGLIST(S),                                       835
         WHILE SA REPEAT DO                                                  836
            IF VARIABLE(HD SA) THEN DO                                       837
               IF CODEQ(J=GETPROP(PREDSYM(S),HD NFN)) THEN DO                838
                  V= J(PRED),                                                839
                  IF V NE PRED THEN DO                                       840
                     BIND(HD SA,V), RETURN TRUE                              841
                  END                                                        842
               END                                                           843
            END,                                                             844
            NFN=TL NFN,SA= TL SA                                             845
         END,                                                                846
         MERITVAL(CONTEXT)=MERIT, RETURN S                                   847
      END                                                                    848
   END;                                                                      849
GROUND=  PROC(X),                                                            850
      BEGIN(T),                                                              851
         IF ATOM(X) THEN DO                                                  852
            (IF VARIABLE(X) THEN RETURN FALSE),                              853
            RETURN TRUE                                                      854
         END,                                                                855
         T= TL X,                                                            856
         WHILE T REPEAT DO                                                   857
            IF ¬GROUND(HD T) THEN RETURN FALSE, T= TL T                      858
         END,                                                                859
         RETURN TRUE                                                         860
      END                                                                    861
   END;                                                                      862
HASCON=  PROC(LIT),                                                          863
      BEGIN(),                                                               864
```

```
        IF ATOM(LIT) THEN RETURN FALSE,                                    865
        IF ¬ TL LIT THEN RETURN TRUE,                                      866
        LIT= TL LIT,                                                       867
        WHILE LIT REPEAT DO                                                868
           IF HASCON(HD LIT) THEN RETURN TRUE,LIT=TL LIT                   869
        END,                                                               870
        RETURN FALSE                                                       871
     END                                                                   872
  END;                                                                     873
FREEVARS= PROC(LITS),                                                      874
     BEGIN(L,H,S),                                                         875
        S=0,L=NIL,H=LITS,                                                  876
        WHILE H REPEAT DO                                                  877
           GETFREE(HD H), H= TL H                                          878
        END, RETURN S                                                      879
     END                                                                   880
  END;                                                                     881
GETFREE= PROC(LIT),                                                        882
     BEGIN(T),                                                             883
        IF ATOM(LIT) THEN DO                                               884
           IF ¬L THEN DO                                                   885
              L= LIT:NIL,S=1,RETURN NIL                                    886
           END,                                                            887
           IF MEMBER(LIT ,L) THEN RETURN NIL,                              888
           L=LIT:L,S=S+1,RETURN NIL                                        889
        END,                                                               890
        T= TL LIT,                                                         891
        WHILE T REPEAT DO                                                  892
           GETFREE(HD T), T= TL T                                          893
        END, RETURN NIL                                                    894
     END                                                                   895
  END ;                                                                    896
*                                                                          897
*          COMPLETIONS                                                     898
*                                                                          899
ADDMIXED=PROC(NEWCL),                                                      900
     MIXEDCL=NEWCL:MIXEDCL                                                 901
  END;                                                                     902
*   CL IS A LIST OF LITERALS                                               903
* WE REMOVE EACH LITERAL EQUAL TO LIT AND REPLACE IT BY  A                 904
* ONE ARGUMENT PREDICATE =MARK THE LITERAL BECOMES THE ARGUMENT            905
* LATER THE CLAUSE WILL BE CHECKED FOR COMPLETIONS                         906
* IF A MARKED LITERAL IS AT THE TOP OR TWO MARKED LITERALS ARE             907
* TOGETHER THEN THEY MAY BE COMPLETED                                      908
MARK=    PROC(CL,LIT),                                                     909
     BEGIN(B),                                                             910
        B= CL,                                                             911
        WHILE CL REPEAT DO                                                 912
           IF EQUAL(HD CL,LIT) THEN HD CL= LIST(=MARK,LIT),                913
           CL= TL CL                                                       914
        END , RETURN B                                                     915
     END                                                                   916
  END;                                                                     917
* MERGE REMOVES MARKED LITERALS FROM THE TOP OF A LIST                     918
```

```
* IT DOES NOT CHECK FOR MERGES                                          919
* HOWEVER THIS IS JUST THE PLACE TO PUT IN THAT CHECK                   920
MERGE=   PROC(CL),                                                     921
     BEGIN( ),                                                         922
        WHILE HD HD CL EQ =MARK REPEAT CL = TL CL, RETURN CL           923
     END                                                               924
  END;                                                                 925
* MOVE FINDS THE FIRST NON MARKED LITERAL OF A LIST                     926
MOVE=    PROC(LIST),                                                   927
     BEGIN(),                                                          928
        WHILE LIST REPEAT DO                                           929
           IF HD HD LIST NE =MARK THEN RETURN LIST,                    930
           LIST= TL LIST                                               931
        END, RETURN NIL                                                932
     END                                                               933
  END;                                                                 934
** TEST FOR COMPLETIONS MARKED LITS AT TOP OR NEXT TO EACHOTHER        935
TESTCOMP=PROC(CL),                                                     936
     BEGIN(SWITCH),                                                    937
        SWITCH= TRUE,                                                  938
        WHILE CL REPEAT DO                                             939
           IF HD HD CL EQ =MARK THEN                                   940
           ( IF SWITCH THEN COMPLETR(HD CL) ELSE SWITCH = TRUE )       941
           ELSE SWITCH = FALSE, CL= TL CL                              942
        END                                                            943
     END                                                               944
  END;                                                                 945
COMPLETR=PROC(LIT),                                                    946
     BEGIN(S,J,TEMP,G,T2,L,N,NEGLITS,UN,U,CL),                         947
        S=PREDSYM(LIT),                                                948
        IF(¬(J=CODEQ(GETPROP(S,=CMPLTR))))THEN RETURN NIL,             949
        TEMP=J(LIT),                                                   950
        WHILE TEMP REPEAT DO                                           951
           IF HD TEMP EQ TRUE THEN DO                                  952
              G= GENCLAUS(GENSYM(),NIL,LIST(LIT)),                     953
              ADDMIXED(G)                                              954
           END                                                         955
           ELSE DO                                                     956
              T2= GETPROP(HD TEMP,=DESEND): (HD TEMP),                 957
              WHILE(=(CL.T2)= T2) REPEAT DO                            958
                 NEGLITS=LITLIST(CL),                                  959
                 WHILE =(N.NEGLIT)=NEGLITS REPEAT DO                   960
                    UN= MGUSET(N,COPY(LIT), MAXCOERS+1,0),             961
                    WHILE=(U.UN)=UN REPEAT DO                          962
                       ADDMIXED(GENCLAUS(G=GENSYM(),SUBINPLA(POSLIT OF 963
                       CL,HD U),                                       964
                       SUBTRACT(LITLIST(CL), LIT,HD U))),              965
                       IF CLENGTH(G) EQ 0 THEN COMPLETR (G)            966
                    END                                                967
                 END                                                   968
              END                                                      969
           END,                                                        970
           TEMP= TL TEMP                                               971
        END,RETURN TRUE                                                972
```

```
      END                                                                    973
   END;                                                                      974
**                                                                          975
**              RUNTIME MONITOR AND INITAIZATION RTNS                       976
**                                                                          977
PRINTS=  PROC(),                                                            978
      BEGIN(),                                                              979
         S= COERS, PRINT(≠COERCIONS≠),                                      980
         WHILE S REPEAT DO                                                  981
            J= VALUE (HD S),                                                982
            PRINT(HD S, LITLIST(J), GETPROP(HD S,=PAIR)), S= TL S           983
         END,                                                               984
         S= MIXEDCL, PRINT(≠MIXEDCL + POSITIVE CLAUSES≠),                   985
         WHILE S REPEAT DO                                                  986
            PRINT(J= NAMEOF(HD S), GETPROP(J,=POSLIT),≠FROM≠,LITLIST(HD     987
            S)), S= TL S                                                    988
         END,                                                               989
         RETURN TRUE                                                        990
      END                                                                   991
   END;                                                                     992
OTTER=   PROC(X),                                                           993
      BEGIN(S,STIME),                                                       994
         ANDSW= FALSE,                                                      995
         PRINTS(),                                                          996
         X= REMCO(X),                                                       997
         IF ATOM(HD X) THEN S= LIST(LIST(X))                                998
         ELSE IF ATOM(HD HD X) THEN DO                                      999
            IF ANDSW THEN S= LIST(X) ELSE DO                               1000
               S= X, WHILE S REPEAT DO                                     1001
                  HD S=LIST(HD S), S= TL S                                 1002
               END , S= LIST(X)                                            1003
            END                                                            1004
         END                                                               1005
         ELSE S= X,                                                        1006
         NEGS= NIL,                                                        1007
         WHILE S REPEAT DO                                                 1008
            NEGS= GENCLAUS(GENSYM(),NIL,HD S) :NEGS, S= TL S               1009
         END,                                                              1010
         CLEARSYS(),                                                       1011
         CLEARSYS=NIL,                                                     1012
         GARBCOLL(),                                                       1013
         COPY=COPX,                                                        1014
         PRINT(≠START OF RUN≠,STIME=TIME()),                               1015
         S= RTM(NEGS,MIXEDCL,COERS),                                       1016
         PRINTHIS(S),PRINT(TIME()-STIME),PRINT(COPYFAIL),PRINT(≠EOJ≠),     1017
         PRINT(≠RESOLUTIONS≠,RCNTR,≠RESOLVENTS GENERATED≠,RGEN,≠ACCEPTED   1018
         ≠,RACCPT)                                                         1019
         ,                                                                 1020
         PRINT(≠DUPLICATES≠,DCNTR,≠COMPLEX≠,CNTR,≠DEEP≠,DEEPCNTR),         1021
         STOP()                                                            1022
      END                                                                  1023
   END;                                                                    1024
CHGCHAR(=.,10);                                                            1025
CLEARSYS=PROC(),                                                           1026
```

```
      BEGIN(),                                                                  1027
         GARBCOLL(),                                                            1028
         BALM= EXECUTE= VFROML.=NIL,                                            1029
         ORDINAL.=ORD1.=IFROMID.=INITIAT.=INITIO.=INITUNA.=NIL,                 1030
         UNARYLI.=INITINF.=INFIXLI.=INITEXP.=MACROLI.=NIL,                      1031
         INITCOD.=CONGENL.=INITOPL.=OPLIST.=INITMIS.=NIL,                       1032
         READ.=RDTOKEN.=RDITEM.=GETLIST.=GETVECT.=GETV.=NIL,                    1033
         LXSCAN.=READIN.=TRANSLA.=MACDEF.=MACROLI.=INFIX=UNARY=NIL,             1034
         BRACKET= FNOTN.=GETNEXT.=ANALYSE.=REMCOM.=REMSEP. =NIL,                1035
         OPERROR.=EXPAND.=EXLIS.=CODEGEN.=SUBLBLS.=NIL,                         1036
         COMP.= GVAR.= GCON.=GCALLS.= ARGLIST.= GREFS.= ASS.=LBL.=              1037
         GENLBL.=                                                               1038
         GLAMBDA=EXCH1.=EXCH2.= GPROP=SAVL1.=SAVL2.=RESTL1.=RESTL2.=NIL,        1039
         GRETURN= GPROGN=GGO=GCOND= GELSE.=GTHEN.=GAND=GOR=NIL,                 1040
         GQUOTE=GSETQ=ASSIGN.=GWHILE=GFOR=CGCHECK.= MAKPROPS=PRTP.=NIL,         1041
         SETPROPL(= INFIX,NIL),SETPROPL(=UNARY,NIL),                            1042
         SETPROPL(=MACRO,NIL), SETPROPL(=CODEG,NIL),SETPROPL(=INSTR,NIL)        1043
         ,                                                                      1044
         SETPROPL(=INFIXLI.,NIL),SETPROPL(=UNARYLI.,NIL),                       1045
         SETPROPL( =MACROLI.,NIL),SETPROPL(=CODGENL.,NIL)                       1046
         , SETPROPL(=OPLIST.,NIL), REMOVEX.=MMEANS=MMATCH.=TMAC.=MATCH.=        1047
         NIL,                                                                   1048
         BUILD.=GENNAM.=ADDON=CODEGEN=COMPILE=IFROMID=MACDEF=ORDINAL=NIL        1049
         ,                                                                      1050
         RDTOKEN=READ=TRANSLAT= MGO.=MEQUAL.=NIL,                               1051
         CHAR.=NIL,                                                             1052
      END                                                                       1053
   END;                                                                         1054
CHGCHAR(=.,11);                                                                 1055
RTM=     PROC(NEGS,MIXEDCL,COERS),                                              1056
      BEGIN(FINISHED,N,CURCONTE,NEGLIT,NEGCL,M,USET,CL,REDO,MSET,M,             1057
         RETAIN,                                                                1058
         NUMSOLUT,UNIFDEPT),                                                    1059
         DETAIL(NEGS),DETAIL(FINISHED),                                         1060
         WHILE =(N.NEGS)=NEGS REPEAT DO                                         1061
            DETAIL(N), DETAIL(DONE),                                            1062
            IF SAVER THEN DO                                                    1063
               IF SAVER(N) THEN DONE = N: DONE                                  1064
            END ELSE DONE= N:DONE,                                              1065
            NEGCL=LITLIST(N), NEGLIT=SELECT1(NEGCL), REDO= FALSE,               1066
            DETAIL(NEGCL),DETAIL(NEGLIT),                                       1067
            NEGCL=COPY(NEGCL),                                                  1068
            IF NEGPRINT THEN                                                    1069
            PRINT(≠NEG CLAUSE IS≠,NAMEOF(N),≠LITERAL IS≠,NEGLIT),               1070
            NUMSOLUT=0,                                                         1071
            RCNTR= RCNTR+1,                                                     1072
            MSET=LOOKUP(HD NEGLIT,PREDCHAI),                                    1073
            WHILE =(M.MSET)=MSET REPEAT DO                                      1074
               FINISHED=MAXLITS-CLENGTH(M)-CLENGTH(N)+1,                        1075
               IF FINISHED GE 0 THEN DO                                         1076
                  POS= POSLIT OF(NAMEOF(M)),                                    1077
                  DETAIL(N),DETAIL(M),DETAIL(POS),                              1078
                  UNIFDEPT=0,                                                   1079
                  PENDING=NIL,CORID=100,                                        1080
```

```
      BYPASS= NIL,                                                    1081
      IF((USET=MGUSET(COPY(NEGLIT),POS,0,FINISHED,NIL)) NE =          1082
      NOGOOD) THEN DO                                                 1083
         IF CLAUSEPR THEN                                             1084
         PRINT(≠RESOLVE WITH≠,NAMEOF(M)),                             1085
         WHILE =(U.USET)= USET REPEAT DO                              1086
            DETAIL(U),DETAIL(USET),                                   1087
            RETAIN= TRUE, CL= GENNEG(N,M,COPY(NEGLIT),U,              1088
            NEGCL),                                                   1089
            DETAIL(CL),                                               1090
            RGEN=RGEN+1,                                              1091
            RETAIN= DELATE(CL, RETAIN),                               1092
            IF EMPTYCLA(CL) THEN RETURN CL ELSE DO                    1093
               IF RETAIN THEN(IF CLMEMBER(CL,DONE) THEN               1094
               RETAIN= FALSE ELSE RETAIN=                             1095
               ¬CLMEMBER(CL,NEGS)),                                   1096
               IF RETAIN THEN ADDNEGS(CL)                             1097
               , IF CLAUSEPR AND RETAIN THEN DO                       1098
                  PRINT(≠RESOLVENT IS≠,NAMEOF(CL),LITLIST(CL          1099
                  )),                                                 1100
                  PRINT(≠MERIT VALUE IS≠,MERITVAL(CL)),               1101
                  PRINT(≠POOL SIZE IS ≠,LENGTH(NEGS))                 1102
               END                                                    1103
            END                                                       1104
         END                                                          1105
      END                                                             1106
   END                                                                1107
END,                                                                  1108
IF (HD NEGLIT EQ EQ1) OR (HD NEGLIT EQ EQ2) THEN DO                   1109
   RCNTR=RCNTR+1,                                                     1110
   BYPASS= (HD NEGLIT EQ EQ2),                                        1111
   M=REFLEX,                                                          1112
   FINISHED=MAXLITS-CLENGTH(N)+1,                                     1113
   DETAIL(FINISHED) ,                                                 1114
   UNIFDEPT=1,PENDING=NIL,CORID=100,                                  1115
   IF(USET=MGUSET(COPY(HD TL NEGLIT),COPY(HD TL TL NEGLIT),0          1116
   ,FINISHED,NIL)                                                     1117
   ) NE =NOGOOD THEN DO                                               1118
      DETAIL(USET),                                                   1119
      IF CLAUSEPR THEN                                                1120
      PRINT(≠RESOLVE WITH≠,NAMEOF(M)),                                1121
      WHILE =(U.USET)=USET REPEAT DO                                  1122
         DETAIL(U),                                                   1123
         RETAIN=TRUE,CL=GENNEG(N,M,COPY(NEGLIT),U,NEGCL),             1124
         RGEN=RGEN+1,                                                 1125
         RETAIN=DELATE(CL,RETAIN),                                    1126
         IF EMPTYCLA(CL) THEN RETURN(CL)ELSE DO                       1127
            IF RETAIN THEN (IF CLMEMBER(CL,DONE) THEN RETAIN          1128
            = FALSE ELSE                                              1129
            RETAIN=¬CLMEMBER(CL,NEGS)),                               1130
            IF RETAIN THEN ADDNEGS(CL)                                1131
            , IF CLAUSEPR AND RETAIN THEN DO                          1132
               IF CLAUSEPR THEN                                       1133
               PRINT(≠RESOLVENT IS≠,NAMEOF(CL),LITLIST(CL)),          1134
```

```
                        PRINT(≠POOL SIZE IS ≠,LENGTH(NEGS))                    1135
                     END                                                       1136
                  END                                                          1137
               END                                                             1138
            END                                                               1139
         END,                                                                 1140
         IF (HD NEGLIT EQ EQ1) OR (HD NEGLIT EQ EQ2) THEN DO                  1141
            BYPASS= NIL,                                                      1142
            M= TRANS,                                                         1143
            FINISHED=MAXLITS-CLENGTH(N)-1,                                    1144
            IF FINISHED GE 0 THEN DO                                          1145
               IF CLAUSEPR THEN                                               1146
               U= NIL:LIST( LIST(EQ2, COPY(HD TL NEGLIT),=ZZZ),               1147
               LIST( EQ2, =ZZZ,COPY(HD TL TL NEGLIT))),                       1148
               PRINT(≠RESOLVE WITH≠,NAMEOF(M)),                               1149
               RETAIN=TRUE,                                                   1150
               CL= GENNEG(N,M,COPY(NEGLIT),U,NEGCL),                          1151
               RGEN= RGEN+1,                                                  1152
               IF CLAUSEPR THEN                                               1153
               PRINT(≠RESOLVENT IS≠,NAMEOF(CL),LITLIST(CL)),                  1154
               RETAIN= DELATE(CL,RETAIN),                                     1155
               IF CLAUSEPR THEN                                               1156
               PRINT(≠POOL SIZE IS≠,LENGTH(NEGS)),                            1157
               IF RETAIN THEN (IF CLMEMBER(CL,DONE) THEN RETAIN=              1158
               FALSE ELSE RETAIN=                                             1159
               ¬CLMEMBER(CL,NEGS)),                                           1160
               IF RETAIN THEN ADDNEGS(CL)                                     1161
            END                                                               1162
         END,                                                                 1163
         IF REDO THEN ADDNEGS(N)                                              1164
      END                                                                     1165
   END                                                                        1166
  END;                                                                        1167
COMPLEX= PROC(X),                                                             1168
   0                                                                          1169
  END;                                                                        1170
ONCE=    PROC(LIT),                                                           1171
   BEGIN(J,K),                                                                1172
      J=NAMEOF(MIXEDPAR(CURCONTE)),K=NEGPAR(CURCONTE),                        1173
      WHILE(¬INPUTCL(K)) REPEAT DO                                            1174
         IF(NAMEOF(MIXEDPAR(K)) EQ J) THEN RETURN FALSE,                      1175
         K= NEGPAR(K)                                                         1176
      END,                                                                    1177
      RETURN TRUE                                                             1178
   END                                                                        1179
  END;                                                                        1180
ONCE PARTEVAL ONCE;                                                           1181
UNIQ =   PROC(N,P,UOUT),                                                      1182
   BEGIN(),                                                                   1183
      RETURN UOUT EQ NIL                                                      1184
   END                                                                        1185
  END;                                                                        1186
SYM1=    PROC(N,P,UOUT),                                                      1187
   BEGIN(),                                                                   1188
```

```
        IF GROUND(N) OR GROUND(P) THEN RETURN                                    1189
        TRUE ELSE RETURN FALSE                                                   1190
      END                                                                        1191
  END;                                                                           1192
DOWN30P =PROC(PRED),                                                             1193
      BEGIN(),                                                                   1194
        MERIT=MERIT+30, RETURN TRUE                                              1195
      END                                                                        1196
  END;                                                                           1197
DOWN30 PARTEVAL DOWN30P;                                                         1198
PREDCHAI= NIL;                                                                   1199
GENCLAUS(=REFLEX,=(EQ1 X X), NIL);                                               1200
GENCLAUS(=TRANS,NIL,NIL);                                                        1201
SAVER=NIL;                                                                       1202
SELECT1= PROC(LITS),                                                             1203
      BEGIN(Y),                                                                  1204
        Y=LITS, WHILE Y REPEAT DO                                                1205
          IF HD HD Y EQ =MARK THEN Y= TL Y ELSEIF                                1206
          (GETPROP(HD HD Y ,=PE) NE NIL) THEN Y= TL Y ELSE RETURN HD Y           1207
        END,                                                                     1208
        PRINT(≠NO LITERAL MAY BE SELCTED≠,LITS),                                 1209
        RETURN HD Y                                                              1210
      END                                                                        1211
  END;                                                                           1212
COERS=NIL;                                                                       1213
RCNTR= RGEN= RACCPT =0;                                                          1214
NOTEQLIS=NIL;                                                                    1215
COMPLET = PEFNS= NIL;                                                            1216
DONE=NIL;                                                                        1217
MIXEDCL= NIL;                                                                    1218
COPYFAIL= 0;                                                                     1219
BVARS= TRUE;                                                                     1220
CLAUSEPR= TRUE;                                                                  1221
CORID= 0;                                                                        1222
DEBUG=FALSE;                                                                     1223
EQ1= =EQ1; EQ2= =EQ2;                                                            1224
CLAUSEID=1;                                                                      1225
DONE=NIL;MAXLEVEL=10;                                                            1226
BYPASS= NIL; MAXCOERS=1; NOTEQLIS=NIL;                                           1227
MAXCPLEX = 6;                                                                    1228
MAXCPX= 65;                                                                      1229
MAXNEST=7;                                                                       1230
NEGPRINT = TRUE;                                                                 1231
NOSIMPLE = FALSE;                                                                1232
TRACE=1;                                                                         1233
MAXLEVEL=100;                                                                    1234
MAXLITS=10;                                                                      1235
DO                                                                               1236
   SAVEBALM(≠DILEMMA≠),PRINT(=DILEMMA,=JUNE,76)                                  1237
END;                                                                             1238
CLOSE(≠DILEMMA≠);                                                                1239
```

```
*                                                                          1
*           APPENDIX II.                                                   2
*                                                                          3
*      GEOMETRY PROGRAM LISTING                                            4
*                                                                          5
*   SYSTEM UTILITIES AND SWITCHES                                          6
ONEWAY= TRUE;                                                             7
ALWAYS= FALSE;                                                            8
CONEF=IFROMID;                                                            9
MAXNEST=5;                                                               10
MAXLEVEL=15;                                                             11
MAXFREE=4;                                                               12
MAXCPX=120;                                                              13
CONSTANT                                                                 14
   STANG, RTANG, R                                                       15
END;                                                                     16
CONSTANT                                                                 17
   I                                                                     18
END;                                                                     19
CONSTANT                                                                 20
   E,P,K,N,M,C,D,Q,B                                                     21
END;                                                                     22
COEF=CONEF;                                                              23
MAXCOERS=1;                                                              24
POINTQ=  PROC(X),                                                        25
     MEMBER(X,=(E P K N M C D Q B))                                      26
  END;                                                                   27
TYPE XPT = POINTQ;                                                       28
TYPE YPT= POINTQ;                                                        29
TYPE ZPT= POINTQ;                                                        30
TYPE UPT= POINTQ;                                                        31
TYPE VPT= POINTQ;                                                        32
TYPE WPT = POINTQ;                                                       33
TYPEQ=   PROC(X) ,                                                       34
     MEMBER(X,=(R I ))                                                   35
  END;                                                                   36
RQ=      PROC(X),                                                        37
     X EQ =R                                                             38
  END; BQ= PROC(X),                                                      39
     X EQ =I                                                             40
  END;                                                                   41
TYPE TYR = RQ; TYPE TYB = BQ;                                            42
*                                                                        43
* SPECIAL CPROCS OF GEOMETRY                                             44
*                                                                        45
CSYMP=   PROC(N,P,UOUT),                                                 46
     UOUT EQ NIL                                                         47
  END;                                                                   48
LFIPP=   PROC(N,P,UOUT),                                                 49
     BEGIN(),                                                            50
        WORKING =                                                        51
        (                                                                52
        IF VARIABLE(N) THEN FALSE ELSE IF VARIABLE(P) THEN FALSE ELSE    53
        IF GROUND(N) THEN TRUE ELSE IF GROUND(P) THEN TRUE ELSE FALSE    54
```

```
        )                                                                    55
        , IF DEBUG THEN PRINT(WORKING), RETURN WORKING                       56
      END                                                                    57
   END;                                                                      58
LFLIPP CPROC LFLIP;                                                          59
*                                                                            60
*       PARTIAL EVALUATORS OF FUNCTIONS AND PREDICATES                       61
* CANONICAL ORDERINGS ARE DONE HERE                                          62
*                                                                            63
RLINE=   PROC(FN),                                                           64
      BEGIN(X,Y),                                                            65
         CRACK(=(T1 X Y),FN),                                                66
         IF VARIABLE(X) THEN DO                                              67
            IF VARIABLE(Y) THEN DO                                           68
               MERIT=MERIT+10,                                               69
               IF X LT Y THEN RETURN FN,                                     70
               IF X EQ Y THEN DO                                             71
                  RETAIN = FALSE, RETURN FN                                  72
               END,                                                          73
               RETURN LIST(T1,Y,X)                                           74
            END,                                                             75
            RETURN FN                                                        76
         END,                                                                77
         IF VARIABLE(Y) THEN RETURN LIST(T1,Y,X),                            78
         IF(HD X EQ =INTER) THEN RETURN FN,                                  79
         IF(HD Y EQ =INTER) THEN RETURN FN,                                  80
         XPT= HD X, YPT= HD Y,                                               81
         IF XPT EQ YPT THEN DO                                               82
            RETAIN= FALSE, RETURN FN                                         83
         END,                                                                84
         IF CONEF(XPT) LE CONEF(YPT) THEN RETURN FN,                         85
         RETURN LIST(T1,LIST(YPT),LIST(XPT))                                 86
      END                                                                    87
   END;                                                                      88
RANG=    PROC(FN),                                                           89
      BEGIN(T1,X,Y,Z),                                                       90
         CRACK(=(T1 X Y Z),FN),                                              91
         IF EQUAL(X,Y) THEN RETAIN= FALSE,                                   92
         IF EQUAL(X,Z) THEN RETAIN= FALSE,                                   93
         IF EQUAL(Y,Z) THEN RETAIN= FALSE,                                   94
         IF ¬CRACK(=(T1 XPT Y ZPT),FN) THEN RETURN FN,                       95
         IF ( COLINP(FN) EQ TRUE) THEN RETURN LIST(=STANG),                  96
         IF CONEF(XPT) LE CONEF(ZPT) THEN RETURN FN,                         97
         IF ¬PAIRQ(Y) THEN                                                   98
         IF ¬VARIABLE(Y) THEN Y= LIST(Y),                                    99
         RETURN LIST(T1,LIST(ZPT),Y,LIST(XPT))                              100
      END                                                                   101
   END;                                                                     102
L PARTEVAL RLINE; A PARTEVAL RANG;                                          103
MIDPTP=  PROC(FN),                                                          104
      BEGIN(X,Y,T1),                                                        105
         CRACK(=(T1 X Y),FN), IF EQUAL( X,Y) THEN RETAIN= FALSE,            106
         RETURN FN                                                          107
      END                                                                   108
```

```
  END;                                                                    109
MIDPT PARTEVAL MIDPTP;                                                    110
DIFF3=   PROC(PRED),                                                      111
     BEGIN(),                                                             112
        CRACK(=(T1 XPT1 YPT1 ZPT1),PRED) ,                                113
        IF EQUAL(XPT1,YPT1) THEN RETURN FALSE,                            114
        IF EQUAL (XPT1,ZPT1) THEN RETURN FALSE,                           115
        IF EQUAL(ZPT1,YPT1) THEN RETURN FALSE,                            116
        IF CRACK(=(T1 XPT YPT ZPT),PRED) THEN RETURN TRUE ,        THEN DO 117
           IF ¬ MEMBER(XPT,LIST(ZPT,UPT,VPT,WPT )) AND (¬MEMBER(YPT,      119
           LIST(                                                          120
           ZPT,UPT,VPT,WPT))) THEN RETURN TRUE, RETURN FALSE              121
        END,                                                              122
        RETURN PRED                                                       123
     END                                                                  124
  END;                                                                    125
DIFF3 PARTEVAL DIFF3;                                                     126
DOWN1 =  PROC(PRED),                                                      127
     BEGIN(),                                                             128
        IF ¬CRACK(=(T1 XPT YPT ZPT UPT VPT WPT),PRED) THEN MERIT=MERIT+   129
        30,                                                               130
        RETURN TRUE                                                       131
     END                                                                  132
  END;                                                                    133
DOWN PARTEVAL DOWN1;                                                      134
COLINP=  PROC(FN),                                                        135
     BEGIN(Z,S1,S2),                                                      136
        IF CRACK(=(T1 XPT YPT ZPT), FN) THEN DO                           137
           IF MEMB(TL FN,COLINER) THEN RETURN TRUE,                       138
           RETURN FALSE                                                   139
        END,                                                              140
        IF CRACK(=(T1 XPT YPT Z),FN) THEN DO                              141
           IF VARIABLE(Z) THEN DO                                         142
              BIND(Z,LIST(=INTER,=(R),CORID =CORID+1,LIST(XPT),           143
              LIST(YPT),CORID=CORID+1,CORID=CORID+1)),                    144
              RETURN TRUE                                                 145
           END,                                                           146
           IF CRACK(=(T1 TYR T1 UPT VPT S1 S2),Z) THEN DO                 147
              IF (XPT EQ UPT) AND (YPT EQ VPT) THEN RETURN TRUE,          148
              IF VARIABLE(S1) AND VARIABLE(S2) THEN DO                    149
                 BIND(S1,XPT),BIND(S2,YPT),BIND(T1,=(R)),RETURN TRUE      150
              END,                                                        151
              IF ¬ CROSSES(LIST(=INTER,XPT,YPT,UPT,VPT))THEN RETURN       152
              FALSE,                                                      153
              RETURN TRUE                                                 154
           END                                                            155
        END,                                                              156
        IF CRACK(=(T1 Z XPT YPT),FN) THEN RETURN COLINP(LIST(T1,LIST(     157
        YPT),                                                             158
        LIST(XPT),Z)),                                                    159
        RETURN FN                                                         160
     END                                                                  161
  END;                                                                    162
```

```
COLIN PARTEVAL COLINP;                                                    163
CROSSES= PROC(PRED),                                                      164
     BEGIN(T,L,S,J),                                                      165
        IF ¬ CRACK(=(T XPT YPT UPT VPT),                                  166
        PRED) THEN RETURN PRED,                                           167
        IF COEF(XPT) GT COEF(YPT) THEN DO                                 168
           T= XPT,XPT= YPT, YPT= T                                        169
        END,                                                              170
        IF COEF(UPT) GT COEF(VPT) THEN DO                                 171
           T= UPT, UPT= VPT, VPT= T                                       172
        END, S= LIST(XPT,YPT), J= LIST(UPT,VPT),                          173
        L= LOOKSP, WHILE L REPEAT DO                                      174
           IF MEMB(S, HD L) THEN RETURN ¬MEMB (J,HD L), L= TL L           175
        END,                                                              176
        RETURN TRUE                                                       177
     END                                                                  178
  END ; CROSSES PARTEVAL CROSSES;                                         179
INTER=   PROC(FN),                                                        180
     BEGIN(T,S1,S2,S3,S4,TY1,TY2,REP,ITEM),                               181
        IF CRACK(=(T TY1 TY2 XPT YPT UPT VPT),FN) THEN DO                 182
           IF COMEF(XPT) GT COMEF(UPT) THEN DO                            183
              T= XPT, XPT= UPT, UPT= T, T= VPT, VPT=YPT,YPT=T             184
           END,                                                           185
           S1=PNAMES,S2=LIST(XPT,YPT,UPT,VPT),                            186
           WHILE S1 REPEAT DO                                             187
              ITEM=HD HD S1, REP= HD TL HD S1,                            188
              IF EQUAL(S2, TL TL ITEM) THEN DO                            189
                 IF(TY1 EQ HD ITEM) THEN DO                               190
                    IF(TY2 EQ HD TL ITEM) THEN RETURN REP,                191
                    IF VARIABLE(TY2) THEN DO                              192
                       BIND(TY2,HD TL ITEM), RETURN REP                   193
                    END                                                   194
                 END,                                                     195
                 IF(TY2 EQ HD TL ITEM) THEN DO                            196
                    IF VARIABLE(TY1) THEN DO                              197
                       BIND(TY1,HD ITEM), RETURN REP                      198
                    END                                                   199
                 END,                                                     200
                 IF VARIABLE(TY1) AND VARIABLE(TY2) THEN DO               201
                    BIND(TY1,HD ITEM),BIND(TY2,HD TL ITEM),RETURN REP     202
                 END                                                      203
              END,                                                        204
              S1= TL S1                                                   205
           END                                                            206
        END,                                                              207
        IF CRACK(=(T TYR TYR S1 XPT S2 YPT),FN) THEN ( IF XPT EQ YPT      208
        THEN RETURN                                                       209
        YPT),                                                             210
        IF CRACK(=(T TYB TYR XPT S1 S2 YPT),FN) THEN ( IF XPT EQ YPT      211
        THEN                                                              212
        RETURN XPT),                                                      213
        IF CRACK(=(T TYB TYR S1 XPT S2 YPT) ,FN) THEN( IF XPT EQ YPT      214
        THEN                                                              215
        RETURN XPT),                                                      216
```

```
        IF CRACK(=(T TYB TYB S1 S2 S3 S4),FN) THEN DO                          217
           IF PAIRQ(S1) THEN S1= HD S1, IF PAIRQ(S2) THEN S2= HD S2,           218
           IF PAIRQ(S3) THEN S3= HD S3, IF PAIRQ(S4) THEN S4= HD S4,           219
           IF S1 EQ S3 THEN RETURN S1, IF S1= S4 THEN RETURN S1,               220
           IF S2 EQ S3 THEN RETURN S2, IF S2 EQ S4 THEN RETURN S2              221
        END,                                                                   222
        RETURN FN                                                              223
     END                                                                       224
  END;                                                                         225
INTER PARTEVAL INTER;                                                          226
*                                                                              227
*   DILEMMA ASSERTIONS AND THEOREMS                                            228
*                                                                              229
* ANGLE EQUALITY                                                               230
ASYM COER EQANG(X,Y),EQANG(Y,X);                                               231
ANAME COER A(X,Y,Z),A(S,Y,R),COLIN(X,S,Y) ^ COLIN(Z,R,Y);                      232
ANAME1 COER A(X,Y,Z),A(S,Y,Z),COLIN(X,S,Y);                                    233
ANAME2 COER A(X,Y,Z),A(X,Y,R),COLIN(Z,R,Y);                                    234
VERTICAL THEOREM EQANG(A(X,Y,Z),A(U,Y,W)) FROM COLIN(X,Y,W) ^ COLIN(U          235
,Y                                                                             236
,Z)                                                                            237
;                                                                              238
ANGR COER A(X,Y,Z),A(Z,Y,X);                                                   239
UNIQ CPROC ANAME1; UNIQ CPROC ANAME2; UNIQ CPROC ANAME;                        240
LFLIPP CPROC ANGR;                                                             241
ALTER2 THEOREM EQANG(A(Z,Y,X),A(Z,R,S)) FROM PARALLEL(L(X,Y),L(R,S))           242
 ^ COLIN(R,Z,Y);                                                               243
ALTER1 THEOREM EQANG(A(X,Y,Z),A(U,Z,Y)) FROM PARALLEL(L(Y,X),L(U,Z));         244
AREFLX ASSERT EQANG(X,X);                                                      245
ATRANS THEOREM EQANG(X,Y) FROM EQANG(X,Z) ^ EQANG(Z,Y) ^ DOWN3O(X);            246
APART THEOREM EQANG(A(X,U,Z),A(V, R,T)) FROM EQANG (A(X,U,Y),A(V,R,W))        247
 ^ EQANG(A(Y,U,Z),A(W,R,T)) ^ DOWN3O(X);                                       248
ACONG THEOREM EQANG(A(X,Y,Z),A(U,V,W)) FROM CONG(X,Y,Z,U,V,W)                  249
 ^ DIFF3(X,Y,Z) ^ DIFF3(U,V,W);                                                250
ACOR THEOREM EQANG(A(R,Y,X),A(Z,X,J)) FROM PARALLEL(L(W,R),L(U,Z))             251
 ^ COLIN(W,Y,R) ^ COLIN(U,X,Z) ^ COLIN(Y,X,J);                                 252
AISSOC THEOREM EQANG(A(X,Y,Z),A(Y,X,Z)) FROM EQLINE(L(X,Z),L(Y,Z));           253
ATRANS THEOREM EQANG(X,Y) FROM EQANG(X,Z) ^ EQANG(Z,Y) ^ DOWN3O(X);            254
RTEQ THEOREM EQANG(A(X,Y,Z),RTANG) FROM EQANG(A(T,Y,X),RTANG)                  255
 ^ COLIN(T,Y,Z);                                                               256
*SEGMENT EQUALITY                                                              257
LFLIP COER L(X,Y),L(Y,X);                                                      258
LSYM COER EQLINE(X,Y),EQLINE(Y,X);                                             259
LPARBASE THEOREM EQLINE(L(X,Y),L(Y,Z)) FROM                                    260
EQLINE(L(Z,W),L(W,V)) ^ PARALLEL(L(Y,W),L(X,V)) ^ COLIN(X,Y,Z) ^               261
COLIN(Z,W,V) ^ ONCE(X);                                                        262
LCONG THEOREM EQLINE(L(X,Y),L(U,V)) FROM CONG(Y,Z,X,V,W,U) ^ DIFF3(X,          263
Y,                                                                             264
Z)                                                                             265
 ^ DIFF3(U,V,W);                                                               266
SYM1 CPROC LSYM; SYM1 CPROC ASYM;                                              267
LMID ASSERT EQLINE(L(X,MIDPT(X,Y)),L(Y,MIDPT(X,Y)));                           268
LPART THEOREM EQLINE(L(X,Y),L(U,V)) FROM                                       269
EQLINE(L(X,R),L(U,S)) ^ EQLINE(L(R,Y),L(S,V)) ^ COLIN(X,R,Y) ^                 270
```

```
COLIN(U,                                                                271
S,V);                                                                   272
LTRANS THEOREM EQLINE(L(X,Y),L(U,V)) FROM EQLINE(L(X,Y),L(S,T))  ^      273
EQLINE(L(S,T),L(U,V))  ^  DOWN30(X);                                    274
LREF ASSERT EQLINE(X,X);                                                275
LISOC THEOREM EQLINE(L(X,Z),L(Z,Y)) FROM EQANG(A(Z,X,Y),A(X,Y,Z));      276
* PARALLEL LINE THEOREMS                                                277
PARTRANS THEOREM                                                        278
PARALLEL(X,Y) FROM PARALLEL(X,Z)  ^  PARALLEL(Z,Y) ;                    279
PCR1 THEOREM PARALLEL(L(X,INTER(TP1,TP2,U,V,X,Z)),L(X,Z))               280
FROM CROSSES(X,Z,U,V);                                                  281
PARINT THEOREM PARALLEL(L(INTER(T1,T2,X,Y,U,V),V),L(S,T)) FROM          282
PARALLEL(L(S,T),L(U,V));                                                283
MIDPAR THEOREM                                                          284
PARALLEL(L(Y,U),L(Z,V)) FROM                                            285
EQLINE(L(Z,Y),L(Y,X))  ^  COLIN(Z,Y,X)  ^  EQLINE(L(V,U),L(U,X))  ^     286
COLIN(V,U,X) ^ ONCE(X);                                                 287
PCR2 THEOREM PARALLEL(L(X,INTER(TP1,TP2,X,Z,U,V)),L(X,Z)) FROM CROSSES  288
(X,Z,U,V);                                                              289
PEND THEOREM PARALLEL(L(X,Y),L(X,Z)) FROM COLIN(X,Y,Z);                 290
PBASE THEOREM PARALLEL(L(X,Y),L(U,V)) FROM                              291
PARALLEL(L(S,Y),L(U,V))  ^  COLIN(X,S,Y);                               292
PEXT THEOREM PARALLEL(L(X,Y),L(U,V)) FROM                               293
PARALLEL(L(X,Y),L(R,V))  ^  COLIN(R,U,V);                               294
PSYM COER PARALLEL(X,Y), PARALLEL( Y,X);                                295
P1 THEOREM PARALLEL(L(X,Y),L(U,V)) FROM EQANG(A(Y,Z,W),A(V,W,R))  ^      296
COLIN(X,Y,Y)  ^  COLIN(Y,W,V)  ^  COLIN(S,Z,W,R);                       297
P2 THEOREM PARALLEL(L(Z,Y),L(W,V)) FROM                                 298
EQANG(A(Y,Z,W),A(V,W,S))  ^  COLIN(Z,W,S);                              299
* TRIANGLE CONGRUENCE SECTION                                           300
CSYM COER CONG(X,Y,Z,U,V,W),CONG(U,V,W,X,Y,Z);                          301
CASA THEOREM CONG(X,Y,Z,U,V,W) FROM                                     302
EQLINE(L(U,V),L(X,Y))  ^  EQANG(A(X,Y,Z),A(U,V,W))  ^  EQANG(A(Y,X,Z),  303
A(W,U,V));                                                              304
*                                                                       305
CSYMP CPROC CSYM;                                                       306
CSSS THEOREM CONG(X,Y,Z,U,V,W) FROM                                     307
EQLINE(L(X,Y),L(U,V))  ^  EQLINE(L(Y,Z),L(V,W))  ^  EQLINE(L(X,Z),L(U,W 308
));                                                                     309
CSAS THEOREM CONG(X,Y,Z,U,V,W) FROM                                     310
EQLINE(L(X,Y),L(U,V))  ^  EQANG(A(X,Y,Z),A(U,V,W))  ^  EQLINE(L(Y,Z),L( 311
V,W)                                                                    312
);                                                                      313
CPERM THEOREM CONG(X,Y,Z,U,V,W) FROM CONG(Y,Z,X,V,W,U);                 314
CFLIP THEOREM CONG(X,Y,Z,U,V,W) FROM CONG(X,Z,Y,U,W,V);                 315
CPARAL THEOREM CONG(X,Y,Z,X,U,Z) FROM PARALLGM(X,U,Z,Y);                316
CTRANS THEOREM CONG(X,Y,Z,U,V,W) FROM CONG(X,Y,Z,P,Q,R)  ^  CONG(P,      317
Q,R,U,V,W)  ^  DOWN(X,Y,Z,U,V,W);                                       318
CREF ASSERT CONG(X,Y,Z,X,Y,Z);                                          319
*                                                                       320
*      COMPLEXITY MEASSURE AND SELECTION FUNCTIONS                      321
*          THIS SECTION HOLDS THE HEURISTIC INFORMATION                 322
*                                                                       323
GRNDUNIT=NIL;                                                           324
```

```
SELECT1= PROC(LITS),                                                    325
     BEGIN(Y,A),                                                        326
        Y=LITS,                                                         327
        DETAIL(Y),                                                      328
        WHILE Y REPEAT DO                                               329
           IF HD HD Y EQ =PARALLEL THEN                                 330
           DO                                                           331
              A=HD TL HD Y,                                             332
              IF PAIRQ(A) THEN A= HD TL TL A,                           333
              IF PAIRQ(A) THEN A= HD A,                                 334
              DETAIL(A),                                                335
              IF A EQ =INTER THEN RETURN HD Y                           336
           END,                                                         337
* TEST IF HD Y IS (PARALLEL (L X (INTER ...)) ( ) )                     338
           Y= TL Y                                                      339
        END,                                                            340
        Y= LITS, WHILE Y REPEAT DO                                      341
           DETAIL(Y),                                                   342
           IF HD HD Y EQ =MARK THEN Y= TL Y                             343
           ELSEIF GETPROP(HD HD Y ,=PE) THEN Y= TL Y ELSE RETURN HD Y   344
        END,                                                            345
        PRINT(=NO,=LIT,=MAY,=BE,=SELECTED,LITS),                        346
        RETURN HD LITS                                                  347
     END                                                                348
  END;                                                                  349
COMPLEX= PROC(LITS),                                                    350
     BEGIN(S,J,END,HTEMP,FAIL,PCNT,CNT,CODE),                           351
        IF ¬ LITS THEN RETURN 0,                                        352
        IF FREEVARS(LITS) GT MAXFREE THEN RETURN 1000,                  353
        CODE=TRUE,                                                      354
        IF LENGTH (LITS) EQ 1 THEN GRNSAVE= HD LITS,                    355
        PCNT=0,CNT=0,                                                   356
        S=0,                                                            357
        FAIL= TRUE,                                                     358
        IF LENGTH(LITS) LT (MAXLITS/2) THEN FAIL= FALSE,                359
        IF ¬INPUTCL(CURCONTE) THEN DO                                   360
           J=NAMEOF(MIXEDPAR(CURCONTE)),                                361
           IF MEMBER(J,HYP) THEN S= S-50                                362
           ,                                                            363
           IF DEPTHVAL(CURCONTE) GT 6 THEN DO                           364
              J= NAMEOF(MIXEDPAR(NEGPAR(CURCONTE))), IF MEMBER(J,HYP)   365
              THEN S= S-                                                366
              50 ELSEIF MEMBER(NAMEOF(MIXEDPAR(NEGPAR(NEGPAR(CURCONTE )  367
              ))),HYP) THEN                                             368
              S=S-30                                                    369
           END                                                          370
        END,                                                            371
        WHILE LITS REPEAT DO                                            372
           IF ¬((J=HD HD LITS) EQ =MARK) THEN DO                        373
              IF J EQ =PARALLEL THEN PCNT= PCNT +1,                     374
              IF J EQ =COLIN THEN CNT= CNT +1,                          375
              IF ¬(GETPROP(J,=PE) ) THEN DO                             376
                 IF GROUND(HD LITS) THEN DO                             377
                    IF MEMBER(HD LITS, GRNDUNIT) THEN DO                378
```

```
                    PRINT(=SUBSUMED,=BY,=GROUND,=NEGATIVE,=UNIT),          379
                    RETAIN= FALSE,RETURN 1000                              380
                 END                                                       381
              END,                                                         382
              CODE=FALSE,                                                  383
              IF HASCON(HD LITS) THEN DO                                   384
                 FND=FALSE,HTEMP=HYP,                                      385
                 WHILE(¬FND) AND HTEMP REPEAT DO                           386
                    FND= (MGUSET(POSLIT OF (HD HTEMP),COPY(HD LITS),       387
                     0,0,                                                  388
                    NIL) NE                                                389
                    =NOGOOD), IF FND THEN FAIL=FALSE,                      390
                    HTEMP= TL HTEMP                                        391
                 END,                                                      392
                 IF FND THEN S= S-20 ELSE S= S+10                          393
              END , DO                                                     394
                 S= S+2*VARSNUM(HD LITS),                                  395
                 IF J EQ =PARALLEL THEN FAIL=FALSE,                        396
                 IF J EQ =CONG THEN DO                                     397
                    S= S+ 1, FAIL= FALSE                                   398
                 END ELSE S= S+1,                                          399
                 IF ¬GROUND(HD LITS) THEN S=                               400
                 S+ 5                                                      401
              END                                                          402
           END                                                             403
           ELSE DO                                                         404
              S= S+4*VARSNUM(HD LITS) +5                                   405
           END                                                             406
        END , LITS = TL LITS                                               407
     END,                                                                  408
     IF PCNT GE 3 THEN RETURN 1000,                                        409
     IF CODE THEN DO                                                       410
        PRINT(=POSSIBLE,=NULL,=CLAUSE),RETURN 1000                         411
     END,                                                                  412
     IF FAIL THEN RETURN 1000,                                             413
     IF CLENGTH(CURCONTE) EQ 1 THEN DO                                     414
        IF GROUND(GRNSAVE) THEN GRNDUNIT= COPY(GRNSAVE):GRNDUNIT           415
     END,                                                                  416
     RETURN S                                                              417
  END                                                                      418
END;                                                                       419
*                                                                          420
*       THE INPUT FOR THE G5.1 EXAMPLE                                     421
*                                                                          422
* G5 EXAMPLE 6                                                             423
HYP= =(H1 H2 H3);                                                          424
COLINER = =(((E)(P)(K)) ( (K)(P)(E) )((P)(N)(M)) ((M)(N)(P))               425
((K)(M)(D)) ((D)(M)(K))                                                    426
( (E)(B)(D)) ((D)(B)(E))                                                   427
((C)(M)(B))((B )(M)(C))                                                    428
((E)(N)(C)) ((C)(N)(E)) );                                                 429
H1 ASSERT PARALLEL(L(K,C),L(E,D));                                         430
H2 ASSERT EQLINE(L(E,N),L(N,C));                                           431
H3 ASSERT EQLINE(L(K,M),L(M,D));                                           432
```

```
H4 ASSERT PARALLEL(L(P,M),L(N,M));                                              433
HYP = =(H1 H2 H3 H4 );                                                          434
LOOKSP==(((K C)(P N) (N M) (P M) (E D)));                                       435
PNAMES==((( R R K P D E ) E ) (( R R C N D E ) E) ((I R E K M N ) P)             436
((I I E C P M) N) ((R R E D K M) D ) );                                         437
MEMBER= MEMB;                                                                   438
PROVE(EQLINE(L(E,P),L(P,K)));                                                   439
```

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